Quenching rate for a nonlocal problem arising in the micro-electro mechanical system [1]

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**Abstract**

In this paper, we study the quenching rate of the solution for a nonlocal parabolic problem which arises in the study of the micro-electro mechanical system. This question is equivalent to the stabilization of the solution to the transformed problem in self-similar variables. First, some a priori estimates are provided. In order to construct a Lyapunov function, due to the lack of time monotonicity property, we then derive some very useful and challenging estimates by a delicate analysis. Finally, with this Lyapunov function, we prove that the quenching rate is self-similar which is the same as the problem without the nonlocal term, except the constant limit depends on the solution itself.

*Keywords:* quenching, micro-electro mechanical system (MEMS), Lyapunov function, non-local, self-similar, asymptotic.

## 1. Introduction

In this paper, we consider the following initial boundary problem

$$u_t = u_{xx} - g(t; u, \lambda)u^{-2}, \quad -1 < x < 1,\ t > 0, \tag{1.1}$$

$$u(\pm 1, t) = 1, \quad t > 0, \tag{1.2}$$

$$u(x, 0) = u_0(x), \quad x \in [-1, 1], \tag{1.3}$$

where

$$g(t; u, \lambda) := \lambda\Big(1 + \int_{-1}^{1} u^{-1}(\xi, t)d\xi\Big)^{-2}. \tag{1.4}$$

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Throughout this paper, we always assume that

$$\begin{gathered} u_0 \text{ is smooth}, \quad u_0(\pm 1) = 1, \quad 0 < u_0(x) \leqslant 1, \quad u_0(x) = u_0(-x), \\ u_0'(x) \geqslant 0, \quad u_0''(x) \geqslant 0 \quad \text{for } 0 \leqslant x \leqslant 1. \end{gathered} \tag{1.5}$$

Also, we shall simply denote $g(t; u, \lambda)$ by $g(t)$ when there is no confusion.

The problem (1.1)-(1.3) arises in the study of the micro-electro mechanical system. We refer to [27, 28] for the physical background of this model. In fact, equation (1.1) is a special case of the following general model

$$\varepsilon u_{tt} + u_t = \Delta u - \frac{\lambda f(x)}{u^2 \left(1 + \alpha \displaystyle\int_\Omega u^{-1}(\xi, t) d\xi\right)^2}, \quad x \in \Omega, \ t > 0, \tag{1.6}$$

where $u$ represents the distance of the membrane and the ground electrode plate, $\varepsilon$ is the ratio of the interaction due to the inertial and damping terms, $\lambda$ is the applied voltage, $\alpha \geqslant 0$ is related to the capacitor and $f(x)$ is the varying dielectric properties of the membrane. The model (1.6) has been studied extensively, see, e.g., [20, 7, 8, 9, 17, 21, 22, 23, 25, 19] for the case $\varepsilon = 0$ (without inertia) and [24, 18] for the case $\varepsilon > 0$. We also refer the reader to a recent survey paper [16] for more details and some open problems.

It is known [17] that

**Theorem 1.** *Let* (1.5) *hold. Then*

*(a), the system* (1.1)–(1.3) *admits a unique classical solution in the maximal existence interval* $[0, T)$*, i.e., for any small* $\delta > 0$*, the solution is in the class* $u \in C^{2+\alpha,(2+\alpha)/2}([-1, 1] \times [0, T - \delta])$, $\min_{|x| \leqslant 1, 0 \leqslant t \leqslant T - \delta} u(x, t) > 0$*; furthermore, either* $T = \infty$*, or* $0 < T < \infty$.

*(b), for* $\lambda$ *suitably large, the maximal existence interval* $[0, T)$ *is finite, i.e., solution* $u(x, t)$ *of* (1.1)-(1.3) *quenches in finite time* $t = T$*, and* $u(0, t) = \min_{|x| \leqslant 1} u(x, t) \to 0$ *as* $t \to T^-$*. Moreover,* $x = 0$ *is the only quenching point.*

**Remark 1.1.** *It is also clear that*

$$g(t; u, \lambda) u^{-2} \in C^{2+\alpha,(2+\alpha)/2}([-1, 1] \times [0, T - \delta]) \text{ for any small } \delta > 0. \tag{1.7}$$

*In our proofs, we actually derived the Hölder continuity of the solution up to the time* $T$ *(see Lemma 2.3):*

$$u \in C^{1/2,1/4}([-1, 1] \times [0, T]).$$

In engineering application, quenching means the touchdown of membrane to the ground plate. Due to the wide range of applications, there are other studies on problems involving a nonlinear singular term with a negative power and the $p$-Laplace operator. For this, we refer the reader to some recent works [1, 4, 2, 3] and the references cited therein.

The main purpose of this paper is to study the temporal quenching rate of the nonlocal problem (1.1)-(1.3). In fact, the study of temporal singular rates has been one of the important issues in the formation of singularities (such as blow-up, quenching, extinction and dead-core). This can be traced back to the seminal works of Giga and Kohn ([10, 11]) for the study of blow-up rate. Since then, the study of temporal singular rates has attracted a lot of attentions. The temporal singular rates can be either self-similar or non-self-similar. We refer the reader to the references for various temporal singular rates cited in [15].

In particular, for the study of quenching rate for the following equation

$$u_t = u_{xx} - \lambda u^{-p}, \quad p > 0, \tag{1.8}$$

we refer the reader to [12, 5, 13, 26]. It is shown that the quenching rate is self-similar. In other words, the quenching rate for (1.8) is the same as that of the rate for the corresponding spatially independent ordinary differential equation. The quenching profile for (1.8) was studied in [6]. However, little is done for the quenching rate of nonlocal problems. In [14], the quenching rate for the nonlocal equation

$$u_t = u_{xx} - \lambda \left\{ 1 - \int_0^1 u(\xi, t) d\xi \right\} u^{-p}, \quad 0 < x < 1, \ t > 0,$$

was studied for any $p > 0$. It is proved that the quenching rate is self-similar which is the same as that of (1.8).

Throughout this paper, we always assume that $\lambda \gg 1$ such that quenching occurs.

To study the quenching rate, we introduce the following self-similar variables

$$y := x/\sqrt{T-t}, \quad s := -\ln(T-t), \quad z(y,s) := (T-t)^{-1/3} u(x,t). \tag{1.9}$$

Then system (1.1)-(1.3) is equivalent to

$$z_s - z_{yy} + \frac{1}{2} y z_y - \frac{1}{3} z + h(s) z^{-2} = 0, \quad |y| < R(s), \ s > s_0, \tag{1.10}$$

$$z(\pm R(s), s) = e^{s/3}, \quad s > s_0, \tag{1.11}$$

$$z(y, s_0) = T^{-1/3} u_0(T^{1/2} y) \triangleq z_0(y), \quad |y| \leqslant R(s_0), \tag{1.12}$$

where

$$h(s) := g(T - e^{-s}), \quad R(s) := e^{s/2}, \quad s_0 := -\ln T. \tag{1.13}$$

Then the determination of quenching rate is equivalent to the study of the asymptotic behavior of $z(y,s)$ as $s \to \infty$.

A standard method for the study of long time behavior is to construct a suitable Lyapunov function with the help of some a priori estimates. For our equation (1.10)–(1.12), we shall construct this Lyapunov function based on the equation (3.5)–(3.6), which is a semilinear equation of divergence form. It is not hard to find a suitable Lyapunov function for a semilinear parabolic equation. However, the verification of this constructed Lyapunov function relies on certain useful estimates. In particular, the Lyapunov function relies upon the upper and positive lower bounds of the solutions in self-similar variables. In many earlier results such as [12, 5, 13, 14], it is proved that the solution is monotone in time for certain class of initial data. This important property implies the desired upper and positive lower bounds for a Lyapunov function. Unfortunately, in our problem (1.1)-(1.3), this time monotonic property is not available, even if we assume that the solution satisfies the monotonicity property initially.

The lack of time monotonic property is a **grand challenge** for establishing a positive lower bound for the solution in similarity variables, and in this paper we shall derive all the necessary estimates to overcome the difficulties.

In section 3, with a careful construction of various auxiliary functions, we show that the solution in its similarity variables, denoted by $z(y,s)$, will be bounded from below by a positive constant, *provided* an estimate on $\mathscr{F} \triangleq \int_{s_0}^{\infty}\int_{-R(s)}^{R(s)} e^{-y^2/4} z_s^2(y,s)dyds$ is available. Note that the lower bound estimate is crucial because our equation contains terms involving negative powers such as $z^{-1}$ and $z^{-2}$.

The quantity $\mathscr{F}$ appears naturally in the Lyapunov function. In order to obtain estimates on $\mathscr{F}$, we are required to take on the challenge to construct an Lyapunov function without apriorily obtaining estimates on the lower bound of the solution in similarity variables.

We shall achieve this goal in section 4. A key step is to show that $h(s)\frac{d}{ds}K_1(s)$ is integrable over $(s_0,\infty)$, where the function $h(s) = g(T - e^{-s})$ is decomposed as

$$h(s) = \lambda\Big(1 + e^{-s/6}K_1(s) + e^{-s/6}K_2(s)\Big)^{-2},$$

$$K_1(s) := \int_{-R(s)}^{R(s)} e^{-y^2/4} z^{-1}(y,s)dy, \quad K_2(s) := \int_{-R(s)}^{R(s)} [1 - e^{-y^2/4}] z^{-1}(y,s)dy.$$

After differentiation in $s$, $K_1$ involves a possible singularity at $y = 0$ (for lacking a positive lower bound), but $K_2$ contains a factor $y^2$ which will cancel possible singularities at $y = 0$. Directly doing integration by parts will not completely move the $d/ds$ derivatives from $K_1$ to $K_2$. We overcome this difficulty by going to a infinite series, so that we can move *completely* the $d/ds$ derivatives from $K_1$ to $K_2$. Of course, doing so produces other extra terms that need to be estimated, and we are fortunate enough to be able to derive all the estimates necessary for the extra terms.

Since the limit $h(\infty)$ exists (see Lemma 2.1 in §2), the natural candidate for the limit of $z(y,s)$ as $s \to \infty$ is the solution to the problem

$$w_{yy} - \frac{1}{2}yw_y + \frac{1}{3}w - h(\infty)w^{-2} = 0, \quad y \in \mathbb{R}, \tag{1.14}$$

$$w_y(0) = 0 \leqslant w_y(y),\ y \geqslant 0, \text{ and } w \text{ grows at most polynomially at } \infty. \tag{1.15}$$

Our problem corresponds to the case $p = 2$ in (1.8). Since $p = 2 > 1$, there is no slow orbit (see [5]). The only solution to (1.14)–(1.15) with a polynomial bound at $\infty$ is the constant solution $w \equiv w_\infty := [3h(\infty)]^{1/3}$. Note that the constant depends on the solution itself. We now state our main theorem of this paper as follows.

**Theorem 2.** *As $s \to \infty$, the solution $z(y,s)$ converges uniformly on any compact set to the constant $w_\infty$.*

In terms of (1.1)–(1.3), we have the following theorem.

**Theorem 3.** *Let* (1.5) *hold. Suppose that* $u$ *is the solution of* (1.1)–(1.3) *with* $T$ *being the quenching time. Then*

$$\lim_{t\to T^-}\int_{-1}^{1} u^{-1}(\xi,t)d\xi := K \tag{1.16}$$

*exists and is finite. Furthermore, for any* $C > 1$,

$$(T-t)^{-1/3}u(x,t)\to w_\infty \quad \textit{uniformly for } |x|\leqslant C\sqrt{T-t}, \tag{1.17}$$

*where*

$$w_\infty = \Big(3\lambda(1+K)^{-2}\Big)^{1/3}.$$

The rest of this paper is organized as follows. In §2, we provide some a priori estimates for problem (1.1)-(1.3). Then we derive some a priori estimates for the transformed problem (1.10)-(1.12) in §3. In particular, we derive the upper bounds of $z(y,s)$; the lower bounds, however, is derived only for the solution away from $y=0$. Finally, in §4, we first derive some very useful and challenging estimates to construct a Lyapunov function for problem (1.10)-(1.12). With all these estimates in hand, we then prove our main theorem.

## 2. Some a priori estimates for problem (1.1)-(1.3)

In this section, we shall give some a priori estimates for the solution of problem (1.1)-(1.3).

**Lemma 2.1.** *The function* $g(t)$ *is continuous for* $0 \leqslant t \leqslant T$ *and* $\min_{0\leqslant t\leqslant T} g(t) > 0$.

*Proof.* By [17], for any $\beta\in(2,3)$, for sufficiently large $\lambda$ there exists $c>0$ such that

$$c|x|^{2/\beta}\leqslant u(x,t)\leqslant 1 \quad \text{for } |x|\leqslant 1,\ 0\leqslant t\leqslant T. \tag{2.1}$$

It follows that $\int_{-1}^{1}u^{-1}(x,t)dx$ is uniformly integrable and we have the estimate

$$\inf_{0\leqslant t\leqslant T} g(t) > 0.$$

By (2.1), we can apply parabolic estimates in the region $|x|>0$ to obtain

$$\lim_{t\to T^-}u(x,t)=u(x,T) \quad \text{for } |x|>0.$$

The estimate (2.1) also implies $u^{-1}(x,t)\leqslant c^{-1}|x|^{-2/\beta}$. By Lebesgue Dominate Convergence Theorem,

$$\lim_{t\to T^-}\int_{-1}^{1}u^{-1}(x,t)dx=\int_{-1}^{1}u^{-1}(x,T)dx.$$

This shows that $g(t)$ is continuous at $t = T$. $\square$

A careful examination of the proof in [17] shows that the constant $c$ in (2.1) is independent of $\lambda$. Therefore, we have

$$\lim_{\lambda\to\infty} g(T; u, \lambda) = \infty.$$

We next establish

**Lemma 2.2.** *It holds*

$$(T-t)^{-1/3}u(0,t) \leqslant \Big(\frac{3}{T-t}\int_t^T g(\tau)d\tau\Big)^{1/3}. \tag{2.2}$$

*Proof.* Since $u(0,t)$ is the minimum,

$$u_t(0,t) \geqslant -g(t)u^{-2}(0,t).$$

Thus

$$\frac{1}{3}\Big(u^3(0,t)\Big)_t \geqslant -g(t).$$

Recalling that $u(0,T) = 0$, integrating the above inequality leads to (2.2). $\square$

It is not difficult to find (see [17]) that

$$\mathcal{E}(t) \triangleq \frac{1}{2}\int_{-1}^{1} u_x^2(x,t)dx + \lambda\Big(1 + \int_{-1}^{1} u^{-1}(x,t)dx\Big)^{-1}$$

satisfies

$$\mathcal{E}'(t) = -\int_{-1}^{1} u_t^2(x,t)dt < 0.$$

Integrating this equation over $[0,T)$, we obtain $L^2$ estimates for $u_t$:

$$\int_0^T\int_{-1}^{1} u_t^2(x,t)dxdt \leqslant \mathcal{E}(0) < \infty.$$

Using the monotonicity of $\mathcal{E}(t)$, we obtain $L^\infty([0,T],L^2)$ estimate for $u_x$:

$$\frac{1}{2}\int_{-1}^{1} u_x^2(x,t)dx + \lambda\Big(1 + \int_{-1}^{1} u^{-1}(x,t)dx\Big)^{-1} \leqslant \mathcal{E}(0) < \infty.$$

These estimates imply the Hölder continuity of $u$ on $[-1,1]\times[0,T]$ as follows.

**Lemma 2.3.** *The following estimates hold:*

$$|u(x_1,t) - u(x_2,t)| \leqslant C|x_1 - x_2|^{1/2}, \tag{2.3}$$
$$|u(x,t_1) - u(x,t_2)| \leqslant C|t_1 - t_2|^{1/4}. \tag{2.4}$$

*Proof.* The estimate (2.3) is a direct consequence of Hölder's inequality:

$$\begin{aligned} |u(x_1,t)-u(x_2,t)| &\leqslant \Big|\int_{x_1}^{x_2} u_x(\xi,t)d\xi\Big| \leqslant |x_1-x_2|^{1/2}\Big(\int_{-1}^{1} u_x^2(\xi,t)d\xi\Big)^{1/2} \\ &\leqslant \Big(2\mathcal{E}(0)\Big)^{1/2}|x_1-x_2|^{1/2}. \end{aligned}$$

To show (2.4), we note that

$$|u(x,t_1)-u(x,t_2)|^2 \quad\leqslant\quad \Big|\int_{t_1}^{t_2} u_t(x,\tau)d\tau\Big|^2 \leqslant |t_1-t_2|\int_0^T u_t^2(x,\tau)d\tau.$$

Thus for any $x_1 < x_2$, there exists $\bar{x} \in [x_1, x_2]$ such that

$$\begin{aligned} (x_2-x_1)|u(\bar{x},t_1)-u(\bar{x},t_2)|^2 &= \int_{x_1}^{x_2} |u(x,t_1)-u(x,t_2)|^2 dx \\ &\leqslant |t_1-t_2|\int_0^T\int_{-1}^1 u_t^2(x,\tau)dxd\tau, \end{aligned}$$

which implies that

$$|u(\bar{x},t_1)-u(\bar{x},t_2)| \leqslant \frac{|t_1-t_2|^{1/2}}{|x_2-x_1|^{1/2}}(\mathcal{E}(0))^{1/2}.$$

For any $x$, we take $x_1$ and $x_2$ such that $x_1 < x < x_2$, then $|x-\bar{x}| \leqslant |x_2-x_1|$, and

$$\begin{aligned} &|u(x,t_1)-u(x,t_2)| \\ &\leqslant |u(\bar{x},t_1)-u(\bar{x},t_2)| + |u(x,t_1)-u(\bar{x},t_1)| + |u(x,t_2)-u(\bar{x},t_2)| \\ &\leqslant 2\Big(2\mathcal{E}(0)\Big)^{1/2}|x_1-x_2|^{1/2} + \frac{|t_1-t_2|^{1/2}}{|x_2-x_1|^{1/2}}(\mathcal{E}(0))^{1/2}. \end{aligned}$$

Taking $|x_1-x_2| = |t_1-t_2|^{1/2}$, we obtain the desired results. $\square$

As a corollary, we have

**Lemma 2.4.** *It holds*

$$u_x(x,t) \leqslant Cx^{-1/2}.$$

*Proof.* We only need to get an estimate near $x = 0$. By Lemma 2.3, the function

$$\psi(y,s) = \frac{u(ay,a^2s+t^*)-u(0,t^*)}{a^{1/2}}, \qquad 1<y<4, -1<s\leqslant 0, \quad 0<a<1/4,$$

is uniformly bounded. It satisfies

$$\psi_s - \psi_{yy} = -a^{3/2}g(a^2s+t^*)u^{-2}(ay,a^2s+t^*) \triangleq J.$$

By Lemma 2.1 (we choose $\beta$ so that $8/3 < \beta < 3$), the right-hand side of the above equation is estimated by

$$\big|J\big| \leqslant Ca^{3/2}|ay|^{-4/\beta} \leqslant Ca^{3/2-4/\beta} \leqslant C, \quad 1 < y < 4, -1 < s \leqslant 0,\ 0 < a < 1/4.$$

It follows from interior parabolic estimates that

$$\psi_y(y,s) \leqslant C, \qquad 2 \leqslant y \leqslant 3, -1/2 \leqslant s \leqslant 0.$$

Taking $y = 2, s = 0$, we obtain

$$a^{1/2}u_x(2a,t^*) \leqslant C, \qquad 0 < a < 1/4,$$

which implies the conclusion of the lemma. □

We next proceed to establish Hölder continuity of $g(t)$.

**Lemma 2.5.** *There exists $\gamma > 0$ such that*

$$|g(t_1) - g(t_2)| \leqslant C|t_1 - t_2|^\gamma$$

*for any $t_1, t_2 \in [0,T]$.*

*Proof.* It is clear that Hölder continuity of the function $g(t)$ is the same as that for

$$k(t) \triangleq \int_{-1}^{1} u^{-1}(x,t)dx = 2\int_0^1 u^{-1}(x,t)dx.$$

For any $t_1, t_2 \in (0,T]$, $t_1 < t_2$, and any $0 < \mu < 1/4$, we have

$$\begin{aligned}
&|k(t_1) - k(t_2)| \\
\leqslant\ & 2\int_0^{2\mu} |u^{-1}(x,t_1) - u^{-1}(x,t_2)|dx + 2\int_{2\mu}^1 |u^{-1}(x,t_1) - u^{-1}(x,t_2)|dx \\
\triangleq\ & I_1 + I_2.
\end{aligned}$$

It is clear that (note that $c^{-1}$ depends on $\beta$)

$$I_1 \leqslant 4c^{-1}\int_0^{2\mu} |x|^{-2/\beta}dx \leqslant C(\beta)\mu^{1-2/\beta}.$$

To establish an estimate for $I_2$, we compute

$$\begin{aligned}
I_2 \ &= \ 2\int_{2\mu}^1 \frac{|u(x,t_1) - u(x,t_2)|}{u(x,t_1)\cdot u(x,t_2)}dx \\
&\leqslant \ C\mu^{-4/\beta}\int_{2\mu}^1 |u(x,t_1) - u(x,t_2)|dx \ = \ C\mu^{-4/\beta}\int_{2\mu}^1 \Big|\int_{t_1}^{t_2} u_t(x,t)dt\Big|dx \\
&\leqslant \ C\mu^{-4/\beta}|t_1 - t_2|^{1/2}\Big(\int_{2\mu}^1\int_{t_1}^{t_2} u_t^2 dxdt\Big)^{1/2} \\
&\leqslant \ C\mu^{-4/\beta}|t_1 - t_2|^{1/2}.
\end{aligned}$$

Combining the estimates for $I_1$ and $I_2$, we obtain

$$|k(t_1)-k(t_2)| \leqslant C\mu^{1-2/\beta} + C\mu^{-4/\beta}|t_1-t_2|^{1/2}.$$

Taking $\mu = |t_1-t_2|^{\beta/[2(\beta+2)]}$, we obtain

$$|k(t_1)-k(t_2)| \leqslant C|t_1-t_2|^{\gamma}, \quad \gamma := \frac{\beta-2}{2(\beta+2)}.$$

The lemma is proved. □

Since $u$ is clearly bounded by 1 and bounded below by a positive constant near $x=1$, we clearly have the following bounds on the derivatives at $x=1$:

$$0 \leqslant u_x(1,t) \leqslant C, \qquad u_t(1,t)=0, \quad |u_{xx}(1,t)| \leqslant C \quad \text{for } 0 \leqslant t \leqslant T. \tag{2.5}$$

In fact, the estimate (2.1) implies that $u \in C^{1+\alpha,(1+\alpha)/2}([b,1]\times[0,T])$ for any $b \in (0,1)$.

## 3. Some a priori estimates for problem (1.10)-(1.12)

This section is devoted to the derivation of some a priori estimate for the solution $z$ of problem (1.10)-(1.12).

First, in terms of similarity variables, the estimate (2.2) gives

$$z(0,s) \leqslant C_1, \quad \limsup_{s\to\infty} z(0,s) \leqslant \Big(3g(T)\Big)^{1/3} \triangleq \mathscr{H}, \tag{3.1}$$

where

$$C_1 := \sup_{0\leqslant t<T} \Big(\frac{3}{T-t}\int_t^T g(\tau)d\tau\Big)^{1/3}.$$

To obtain an upper bound for $z$, we construct an upper solution as follows. Take $\mu$ and $c_1(\mu)$ such that

$$\frac{2}{3} < \mu < 1, \quad c_1(\mu) = \min_{0<y<\infty}\Big\{(1-\mu)\mu y^{\mu-2} + \Big(\frac{1}{2}\mu - \frac{1}{3}\Big)y^{\mu}\Big\} > 0.$$

Then the function

$$w = y^{\mu} + c_1(\mu)$$

satisfies

$$\begin{aligned} & -w_{yy} + \frac{1}{2}yw_y - \frac{1}{3}w \\ &= (1-\mu)\mu y^{\mu-2} + \Big(\frac{1}{2}\mu - \frac{1}{3}\Big)y^{\mu} - \frac{1}{3}c_1(\mu) \\ &\geqslant \frac{2}{3}c_1(\mu) > 0. \end{aligned}$$

Take $C^*$ such that $C^*c_1(\mu) \geqslant C_1$, $C^*e^{\mu s_0/2} \geqslant e^{s_0/3}$ and $C^*c_1(\mu) \geqslant \|z_0\|_{L^\infty}$. Then it follows from (3.1) and the maximum principle that

$$z(y,s) \leqslant C^*w(y) \quad \text{for } 0 \leqslant y \leqslant e^{s/2},\ s \geqslant s_0.$$

We have established

**Lemma 3.1.** *For any* $\frac{2}{3} < \mu < 1$*, there exists a* $C^* = C^*(\mu) > 0$ *such that*

$$z(y,s) \leqslant C^*(1+|y|^\mu) \quad \text{for } 0 \leqslant y \leqslant e^{s/2},\ s_0 \leqslant s < \infty. \tag{3.2}$$

Recall (2.5). In terms of variables $(z, y, s)$, we have

$$0 \leqslant -z_y(-R(s), s) = z_y(R(s), s) \leqslant CR^{-1/3}(s). \tag{3.3}$$

Differentiate the boundary condition (1.11), we obtain

$$\frac{1}{2} y z_y(y,s) + z_s(y,s) = \frac{1}{3}|y|^{2/3} \quad \text{for } y = \pm R(s).$$

Using (3.3), we conclude

$$|z_s(\pm R(s), s)| \leqslant CR^{2/3}(s). \tag{3.4}$$

To derive a lower bound for $z(y, s)$, we rewrite (1.10) as

$$\rho z_s = \Big(\rho \cdot z_y\Big)_y + \rho \cdot \Big(\frac{1}{3}z - h(s)z^{-2}\Big), \tag{3.5}$$

where

$$\rho(y) := e^{-y^2/4}. \tag{3.6}$$

For any $\alpha \geqslant 0$, we compute

$$\begin{aligned} &\frac{d}{ds}\int_{-R(s)}^{R(s)} \rho(y) z^{1+\alpha}(y,s)dy \\ &\quad = 2\rho(R(s))R'(s)z^{1+\alpha}(R(s),s) + \int_{-R(s)}^{R(s)} \rho(y)(z^{1+\alpha})_s(y,s)dy. \end{aligned} \tag{3.7}$$

It follows from (3.7) with $\alpha = 0$ and (3.5) that

$$\frac{d}{ds}\int_{-R(s)}^{R(s)} \rho(y) z(y,s)dy = J_0 + \int_{-R(s)}^{R(s)} \rho(y)\Big(\frac{1}{3}z - h(s)z^{-2}\Big)dy, \tag{3.8}$$

where

$$J_0 = 2\rho(R(s))R'(s)z(R(s),s) + 2\rho(R(s))z_y(R(s),s).$$

Clearly,

$$|J_0| \leqslant C\exp\Big(-\frac{1}{8}e^{s/2}\Big).$$

And, from Lemma 3.1,

$$\sup_{s>s_0}\int_{-R(s)}^{R(s)} \rho(y) z(y,s)dy < \infty. \tag{3.9}$$

By integrating (3.8), we obtain

$$\Big|\int_{s_1}^{s_2}\int_{-R(s)}^{R(s)} \rho(y)\Big(\frac{1}{3}z - h(s)z^{-2}\Big)dyds\Big| \leqslant C^* < \infty \quad \text{for } s_2 > s_1 \geqslant s_0, \tag{3.10}$$

where $C^*$ is independent of $s_2$ and $s_1$. It follows from (3.9) and (3.10) that

$$\int_{s_1}^{s_1+1}\int_{-R(s)}^{R(s)}\rho(y)z^{-2}(y,s)dyds \leqslant C \quad \text{for any } s_1 > s_0. \tag{3.11}$$

We claim that

$$\int_{s_1}^{s_1+1}\int_{-R(s)}^{R(s)}\rho(y)z^{-2+\alpha}(y,s)dyds \leqslant C \quad \text{for any } s_1 > s_0 \tag{3.12}$$

for any $\alpha > 0$. For $\alpha \geqslant 2$, (3.12) follows from (3.2). For $\alpha \in (0,2)$, it follows from (3.11) and Hölder's inequality with $p = 2/\alpha$ and $q = 1/(1-\alpha/2)$ that

$$\begin{aligned}
&\int_{s_1}^{s_1+1}\int_{-R(s)}^{R(s)}\rho(y)z^{-2+\alpha}(y,s)dyds \\
= \;&\int_{s_1}^{s_1+1}\int_{-R(s)}^{R(s)}\rho^{1/p}(y)\rho^{1/q}(y)z^{-2+\alpha}(y,s)dyds \\
\leqslant \;&\left(\int_{s_1}^{s_1+1}\int_{-R(s)}^{R(s)}\rho(y)dyds\right)^{1/p}\left(\int_{s_1}^{s_1+1}\int_{-R(s)}^{R(s)}\rho(y)z^{-2}(y,s)dyds\right)^{1/q} \\
\leqslant \;& C \quad \text{for any } s_1 > s_0.
\end{aligned}$$

Hence (3.12) holds for any $\alpha > 0$.

Multiplying the equation (3.5) by $(1+\alpha)z^\alpha$, using (3.7) and the above estimates, we obtain

$$\int_{s_1}^{s_1+1}\int_{-R(s)}^{R(s)}\rho(y)\Big\{\Big(z^{(1+\alpha)/2}\Big)_y\Big\}^2 dyds \leqslant C_\alpha \quad \text{for any } s_1 > s_0. \tag{3.13}$$

By Hölder's inequality, for $y > 0$,

$$z^{(1+\alpha)/2}(y,s) - z^{(1+\alpha)/2}(0,s) \leqslant \sqrt{y}\Big\{\int_0^y \Big\{\Big(z^{(1+\alpha)/2}\Big)_y\Big\}^2 dy\Big\}^{1/2}.$$

It follows from (3.13) that

$$\int_{s_1}^{s_1+1} G(s)ds \leqslant C_\alpha, \text{ where } G(s) := \sup_{0<y\leqslant 1}\frac{[z^{(1+\alpha)/2}(y,s) - z^{(1+\alpha)/2}(0,s)]^2}{y}. \tag{3.14}$$

Since $z_y(0,s) = 0$, we have

$$\lim_{y\to 0}\frac{[z^{(1+\alpha)/2}(y,s) - z^{(1+\alpha)/2}(0,s)]^2}{y} = 0.$$

Hence the "sup" in (3.14) is actually achieved and $G(s)$ is a continuous function in $s$.

Combining (3.11) and (3.14), we obtain

$$\int_{s_1}^{s_1+1}\Big\{\sup_{0<y\leqslant 1}\frac{[z^{(1+\alpha)/2}(y,s) - z^{(1+\alpha)/2}(0,s)]^2}{y} + \int_0^{R(s)}\rho(y)z^{-2}(y,s)dy\Big\}ds \leqslant C.$$

By the mean value theorem, for each $s_1 > s_0$, there exists $\tau_1 \in [s_1, s_1+1]$ such that

$$\begin{aligned}
&\sup_{0<y\leqslant 1} \frac{[z^{(1+\alpha)/2}(y,\tau_1) - z^{(1+\alpha)/2}(0,\tau_1)]^2}{y} + \int_0^{R(\tau_1)} \rho(y) z^{-2}(y,\tau_1)dy \\
&= \int_{s_1}^{s_1+1} \Big\{ \sup_{0<y\leqslant 1} \frac{[z^{(1+\alpha)/2}(y,s) - z^{(1+\alpha)/2}(0,s)]^2}{y} + \int_0^{R(\tau_1)} \rho(y) z^{-2}(y,s)dy \Big\} ds \\
&\leqslant C. \qquad (3.15)
\end{aligned}$$

This implies that, for $0 < y \leqslant 1$,

$$z^{(1+\alpha)/2}(y,\tau_1) \leqslant z^{(1+\alpha)/2}(0,\tau_1) + C\sqrt{y},$$

and so (assuming that $\alpha < 1$),

$$z^2(y,\tau_1) \leqslant C[z^{1+\alpha}(0,\tau_1) + y]^{2/(1+\alpha)}.$$

Substituting into (3.15), we obtain

$$\frac{1+\alpha}{1-\alpha} z(0,\tau_1)^{-(1-\alpha)} \leqslant \int_0^1 \frac{dy}{[z^{1+\alpha}(0,\tau_1) + y]^{2/(1+\alpha)}} \leqslant C.$$

Since we take $\alpha < 1$, this implies that

$$z(0,\tau_1) \geqslant c_0$$

for some $c_0 > 0$ independent of $s_1$. Let $k$ to be the first integer bigger than $s_0$, and take $s_1$ to be $k, k+1, k+2, k+3, \cdots$. We have proved:

**Lemma 3.2.** *There exists $\tau_j \in [k+j, k+j+1]$ with $0 \leqslant \tau_{j+1} - \tau_j \leqslant 2$ such that*

$$z(0,\tau_j) \geqslant c_0 \qquad (3.16)$$

*for some positive constant $c_0$ independent of $s_1$.*

Based on Lemma 3.2, we can establish the following lemma on the lower bound of $z$, provided some $L^2$ estimates on $z_s^-$ is available.

**Lemma 3.3.** *Let $c_0$ be given as in* (3.16) *and let*

$$\Lambda = \|h\|_{L^\infty[-\ln T,\infty)}, \quad A = \Lambda\Big(\frac{c_0}{2}\Big)^{-2}, \quad y^* = \sqrt{\frac{c_0}{2A}}.$$

*If*

$$\frac{1}{y^*}\int_{\tau_j}^{\tau}\int_0^{y^*} [(z_s)^-]^2 dyds \leqslant \frac{c_0^2}{32}$$

*for some $\tau \in (\tau_j, \tau_j+2]$, then*

$$z(0,s) \geqslant \frac{c_0}{2} \quad \text{for } \tau_j \leqslant s \leqslant \tau. \qquad (3.17)$$

*Proof.* We already established

$$z(0,\tau_j) \geqslant c_0 > 0, \quad \tau_j \to \infty, \quad 0 < \tau_{j+1} - \tau_j \leqslant 2.$$

Since $y = 0$ is the minimum of $z(y,s)$, we have

$$z(y,\tau_j) \geqslant z(0,\tau_j) \geqslant c_0.$$

We write

$$\mathscr{L}[z] \triangleq z_s - z_{yy} + \frac{1}{2} y z_y - \frac{1}{3} z + h(s) z^{-2} = 0.$$

Since $A = \Lambda(c_0/2)^{-2}$, we have

$$\begin{aligned} \mathscr{L}\Big[\frac{c_0}{2} + \frac{A}{2} y^2\Big] &= -A + \frac{A}{2} y^2 - \frac{1}{3}\Big(\frac{c_0}{2} + \frac{A}{2} y^2\Big) + h(s)\Big(\frac{c_0}{2} + \frac{A}{2} y^2\Big)^{-2} \\ &\leqslant -A + \Lambda\Big(\frac{c_0}{2}\Big)^{-2} = 0 \quad \text{for } |y| < \sqrt{\frac{c_0}{2A}} = y^*. \end{aligned}$$

It is also clear that with our choice of $y^*$ we also have

$$\frac{c_0}{2} + \frac{A}{2} y^2 \leqslant \frac{3c_0}{4} \quad \text{for } |y| \leqslant y^*.$$

By our assumption,

$$\frac{1}{y^*} \int_{\tau_j}^{\tau} \int_0^{y^*} [(z_s)^-]^2 dy ds \leqslant c_0^2/32.$$

By the mean value theorem, there exists $y_1^* \in [0, y^*]$ such that

$$\int_{\tau_j}^{\tau} [(z_s(y_1^*, s))^-]^2 ds \leqslant c_0^2/32.$$

Hence

$$\begin{aligned} -[z(y_1^*, s) - z(y_1^*, \tau_j)] &\leqslant \sqrt{\tau - \tau_j}\Big(\int_{\tau_j}^{\tau} [(z_s(y_1^*, s))^-]^2 ds\Big)^{1/2} \\ &\leqslant \sqrt{2}\sqrt{\frac{c_0^2}{32}} = \frac{c_0}{4} \quad \text{for } \tau_j \leqslant s \leqslant \tau, \end{aligned}$$

which implies

$$z(y_1^*, s) \geqslant z(y_1^*, \tau_j) - \frac{c_0}{4} \geqslant c_0 - \frac{c_0}{4} \geqslant \frac{3c_0}{4} \quad \text{for } \tau_j \leqslant s \leqslant \tau.$$

By symmetry, we also have $z(-y_1^*, s) = z(y_1^*, s)$.

If $y_1^* = 0$, we then already have $z(0,s) \geqslant 3c_0/4$ for $\tau_j \leqslant s \leqslant \tau$. In the case $y_1^* > 0$, we can then apply the comparison principle in the region $\{(y,s);\ |y| < y_1^*, \tau_j < s < \tau\}$ to conclude

$$z(y,s) \geqslant \frac{c_0}{2} + \frac{A}{2} y^2 \quad \text{for } |y| < y_1^*, \tau_j \leqslant s \leqslant \tau.$$

Hence (3.17) follows and this concludes the proof of this lemma. $\square$

**Remark 3.1.** *Lemmas 3.1, 3.2 and 3.3 are valid for any function $h(s)$ bounded by positive constants from above and below, with $c_0$ and $\tau_j$ depending only on these two bounds.*

Next, we show the following lemma which is very useful in the proof of Lemma 4.3 in §4.

**Lemma 3.4.** *There exist constants $L_1 > 0$ and $c_1 > 0$ such that*

$$z(L_1, s) \geqslant c_1 \quad \text{for } s \gg 1. \tag{3.18}$$

*Proof.* Multiplying the equation (3.5) by $z^2$ and integrating, we obtain

$$\frac{d}{ds}\int_{-R(s)}^{R(s)} \frac{1}{3}\rho z^3 dy = -2\int_{-R(s)}^{R(s)} \rho z z_y^2 dy + \int_{-R(s)}^{R(s)} \frac{1}{3}\rho z^3 - h(s)\int_{-R(s)}^{R(s)} \rho dy + J(s),$$

where $J(s)$ comes from various terms generated by integration by parts. By (1.11), (3.2), (3.3) and (3.4), we have

$$|J(s)| \leqslant C\exp\Big(-\frac{1}{8}e^{s/2}\Big).$$

Thus, for any small $\varepsilon > 0$, we can take $S_1 \gg 1$ so that

$$\frac{d}{ds}\int_{-R(s)}^{R(s)} \frac{1}{3}\rho z^3 dy < \int_{-R(s)}^{R(s)} \frac{1}{3}\rho z^3 - (h(\infty) - \varepsilon)\int_{-R(s)}^{R(s)} \rho dy := I(s) \tag{3.19}$$

for $s \gg S_1$. If $I(s^*) < 0$ for some $s^* > S_1$, then (3.19) implies that $I(s) < 0$ for all $s > s^*$. Therefore, $\int_{-R(s)}^{R(s)} \frac{1}{3}\rho z^3$ must reach zero in a finite time, which is a contradiction. Thus we must have

$$\int_{-R(s)}^{R(s)} \frac{1}{3}\rho z^3 \geqslant (h(\infty) - \varepsilon)\int_{-R(s)}^{R(s)} \rho dy \quad \text{for } s > S_1.$$

It follows that

$$\begin{aligned}
\frac{2L_1}{3} z^3(L_1) &\geqslant \int_{-L_1}^{L_1} \frac{1}{3} z^3 dy \\
&\geqslant \int_{-L_1}^{L_1} \frac{1}{3}\rho z^3 dy \\
&\geqslant (h(\infty) - \varepsilon)\int_{-R(s)}^{R(s)} \rho dy - \int_{|y|\geqslant L_1} \frac{1}{3}\rho z^3 dy.
\end{aligned}$$

Since $z \leqslant C(1 + |y|^\mu)$, we can take $L_1$ suitable large so that the right-hand side of the above inequality is uniformly positive, and this implies the conclusion of the lemma. $\square$

## 4. Proof of the main theorem

In this section, we first construct a Lyapunov function. Let

$$E(s) = \frac{1}{2}\int_{-R(s)}^{R(s)} \rho(y) z_y^2(y,s)dy - \frac{1}{6}\int_{-R(s)}^{R(s)} \rho(y) z^2(y,s)dy. \tag{4.1}$$

Then $E(s)$ is bounded from below, due to (3.2). We compute, using (3.5),

$$\begin{aligned}
\frac{d}{ds}E(s) &= \int_{-R(s)}^{R(s)} \rho z_y z_{ys} dy - \frac{1}{3}\int_{-R(s)}^{R(s)} \rho z z_s dy + J_{11}(s) \\
&= -\int_{-R(s)}^{R(s)} (\rho z_y)_y z_s dy - \frac{1}{3}\int_{-R(s)}^{R(s)} \rho z z_s dy + J_{11}(s) + J_{12}(s) \\
&= -\int_{-R(s)}^{R(s)} \rho z_s^2 dy + h(s)\frac{d}{ds}\left\{\int_{-R(s)}^{R(s)} \rho z^{-1} dy\right\} \\
&\qquad + J_{11}(s) + J_{12}(s) + J_{13}(s),
\end{aligned}$$

where

$$\begin{aligned}
J_{11}(s) &:= \rho(R(s)) z_y^2(R(s),s)R'(s) - \frac{1}{3}\rho(R(s)) z^2(R(s),s)R'(s), \\
J_{12}(s) &:= 2\rho(R(s)) z_y(R(s),s) z_s(R(s),s), \\
J_{13}(s) &:= -2h(s)\rho(R(s)) z^{-1}(R(s),s)R'(s).
\end{aligned}$$

Set $J_1(s) := J_{11}(s) + J_{12}(s) + J_{13}(s)$. The we have

$$\frac{d}{ds}E(s) = -\int_{-R(s)}^{R(s)} \rho z_s^2 dy + h(s)\frac{d}{ds}K_1(s) + J_1(s), \tag{4.2}$$

where

$$K_1(s) := \int_{-R(s)}^{R(s)} \rho(y) z^{-1}(y,s)dy. \tag{4.3}$$

Note that, by (3.2), (1.11), (3.3) and (3.4), we have

$$|J_1(s)| \leqslant C\exp\Big(-\frac{1}{8}e^{s/2}\Big). \tag{4.4}$$

Therefore, in order to verify the function $E(s)$ defined by (4.1) is a Lyapunov function, we need to derive some useful estimates for $h(s)$ and the derivative of the function $K_1(s)$ defined by (4.3) to ensure the integrability of $h(s)K_1'(s)$ over $[s_0, \infty)$.

*4.1. Estimates for $h(s)$*

Recall that $h(s) = g(T - e^{-s})$. From the definition of $h(s)$, we derive

$$\begin{aligned}
&\sqrt{\lambda}\, h^{-1/2}(s) - 1 \\
= &\int_{-1}^{1} u^{-1}(x, T - e^{-s})dx = e^{-s/6}\int_{-R(s)}^{R(s)} z^{-1}(y,s)dy \\
= &e^{-s/6}\left\{\int_{-R(s)}^{R(s)} \rho(y)z^{-1}(y,s)dy + \int_{-R(s)}^{R(s)}[1-\rho(y)]z^{-1}(y,s)dy\right\} \\
\triangleq &e^{-s/6}\{K_1(s) + K_2(s)\}.
\end{aligned}$$

The power series $(1+w)^{-1} = \sum_{j=0}^{\infty}(-1)^j w^j$ is absolutely convergent for $|w| < 1$. Therefore we can differentiate this equality in $w$ to obtain

$$(1+w)^{-2} = -\sum_{j=0}^{\infty}(-1)^j j w^{j-1} = \sum_{j=0}^{\infty}(-1)^j (j+1) w^j \quad \text{for } |w| < 1.$$

It follows that

$$\begin{aligned}
\lambda^{-1}h(s) &= (1+e^{-s/6}K_2)^{-2}\left(1 + \frac{e^{-s/6}K_1}{1+e^{-s/6}K_2}\right)^{-2} \\
&= (1+e^{-s/6}K_2)^{-2}\sum_{j=0}^{\infty} b_j\left(\frac{e^{-s/6}K_1}{1+e^{-s/6}K_2}\right)^{j},
\end{aligned} \tag{4.5}$$

where $b_j = (-1)^j(j+1)$, $j \geqslant 0$. The series is uniformly convergent whenever $\left|\frac{e^{-s/6}K_1}{1+e^{-s/6}K_2}\right| < 1$.

We first establish

**Lemma 4.1.** *Let $\beta \in (2,3)$ close to 3 and let*

$$u_* := \lim_{t\to T}\int_{-1}^{1} u^{-1}(x,t)dx > 0.$$

*Then, for $\gamma_1 = \frac{1}{2} - \frac{1}{\beta} > 0$ and $\gamma > 0$ given in Lemma 2.5, we have*

$$|e^{-s/6}K_1(s)| \leqslant Ce^{-\gamma_1 s}, \quad |e^{-s/6}K_2(s) - u_*| \leqslant Ce^{-\gamma s}. \tag{4.6}$$

*for all $s \geqslant s_0$.*

*Proof.* By (2.1), $z(y,s) \geqslant ce^{s/3-s/\beta}|y|^{2/\beta}$, so that

$$e^{-s/6}z^{-1}(y,s) \leqslant c^{-1}e^{-s/6+s/\beta-s/3}|y|^{-2/\beta}, \ \forall\, |y| \leqslant R(s),\ s \geqslant s_0. \tag{4.7}$$

Since $2 < \beta < 3$, we have $-\frac{1}{6} + \frac{1}{\beta} - \frac{1}{3} = -\gamma_1 < 0$. It follows that $|e^{-s/6}K_1(s)| \leqslant Ce^{-\gamma_1 s}$ for all $s \geqslant s_0$. Using this in the relation

$$\int_{-1}^{1} u^{-1}(x, T-e^{-s})dx = e^{-s/6}(K_1 + K_2),$$

applying also Lemma 2.5, we immediately obtain the second estimate in (4.6). This proves the lemma. □

Recall from (3.12) that

$$\int_s^{s+1}\int_{-R(s)}^{R(s)} \rho(y) z^{-k}(y,s)\,dy\,ds \leqslant C$$

for any $s \geqslant s_0$ for any $1 \leqslant k \leqslant 2$. This actually implies

**Lemma 4.2.** *For any $\alpha > 0$*

$$\int_{s_0}^{\infty} e^{-\alpha s} K_1(s)\,ds \leqslant C, \qquad \int_{s_0}^{\infty} e^{-\alpha s} K_1^2(s)\,ds \leqslant C. \tag{4.8}$$

*Proof.* We break the integral into an infinite series:

$$\begin{aligned}
\int_{s_0}^{\infty} e^{-\alpha s} K_1(s)\,ds &= \sum_{j=0}^{\infty} \int_{s_0+j}^{s_0+j+1} e^{-\alpha s} K_1(s)\,ds \\
&\leqslant \sum_{j=0}^{\infty} e^{-\alpha(s_0+j)} \int_{s_0+j}^{s_0+j+1} K_1(s)\,ds \\
&\leqslant C \sum_{j=0}^{\infty} e^{-\alpha(s_0+j)} \leqslant C.
\end{aligned}$$

By Hölder's inequality

$$K_1^2(s) = \Big(\int_{-R(s)}^{R(s)} \rho z^{-1}\,dy\Big)^2 \leqslant C \int_{-R(s)}^{R(s)} \rho z^{-2}\,dy,$$

so that the second inequality can be established in a similar manner. □

*4.2. Estimates for the derivative of $K_2$*

Our key lemma is as follows

**Lemma 4.3.** *There exists $\gamma_2 > 0$ such that*

$$\Big|\frac{d}{ds}\big(e^{-s/6} K_2(s)\big)\Big| \leqslant C e^{-\gamma_2 s}\Big(1 + \int_{-R(s)}^{R(s)} \rho |z_s|\,dy\Big). \tag{4.9}$$

*Proof.* We write $K_2$ as $2\int_0^{R(s)} (1-\rho) z^{-1}\,dy$. Then for any $L > 0$, using (1.11), we have

$$\begin{aligned}
\frac{dK_2}{ds} &= -2\int_0^{R(s)} (1-\rho) z^{-2} z_s\,dy + 2(1-\rho(R(s)) z^{-1}(R(s),s) R'(s) \\
&= 2\int_0^{L} \rho\big(1-\rho^{-1}\big) z^{-2} z_s\,dy + (1-\rho(e^{s/2}))e^{s/6} \\
&\qquad + 2\int_L^{R(s)} \rho z^{-2} z_s\,dy - 2\int_L^{R(s)} z^{-2} z_s\,dy \\
&\triangleq M_1 + M_2 + M_3 + M_4.
\end{aligned}$$

By (2.1), $z^{-2} \leqslant C|y|^{-4/\beta}e^{(2/3-2/\beta)s}$. It is also clear that, for $|y| \leqslant L$, $|1-\rho^{-1}| \leqslant C|y|^2$. It follows that the first term on the right-hand side is estimated by

$$\begin{aligned} |M_1| &= \Big|2\int_0^L \rho\big(1-\rho^{-1}\big)z^{-2}z_s dy\Big| \\ &\leqslant C\int_0^L \rho|y|^{2-4/\beta}e^{(-2/3+2/\beta)s}|z_s|dy \leqslant Ce^{(-2/3+2/\beta)s}\int_0^L \rho|z_s|ds, \end{aligned}$$

so that, if we take $0 < \gamma_2 \leqslant \frac{1}{6} + \Big(\frac{2}{3} - \frac{2}{\beta}\Big)$ (this is possible if we take $\beta$ close to 3),

$$e^{-s/6}|M_1| \leqslant Ce^{-\gamma_2 s}\int_0^L \rho|z_s|ds. \tag{4.10}$$

It is clear that the second term

$$|e^{-s/6}M_2 - 1| = |\rho(e^{s/2})| \leqslant C\exp\Big(-\frac{1}{8}e^{s/2}\Big) \leqslant Ce^{-\gamma_2 s}. \tag{4.11}$$

We assume that $L > L_1$. Since $z(y,s) \geqslant c_0$ for $y > L$, by (3.18) and $z_y \geqslant 0$, the third term is bounded by

$$e^{-s/6}|M_3| = \Big|2e^{-s/6}\int_L^{R(s)} \rho z^{-2}z_s dy\Big| \leqslant Ce^{-s/6}\int_L^{R(s)} \rho|z_s|ds. \tag{4.12}$$

It remains to estimate $M_4$. Using the equation (1.10), we get

$$\begin{aligned} M_4 &= -2\int_L^{R(s)} z^{-2}z_s ds \;=\; 2\int_L^{R(s)} z^{-2}\Big\{-z_{yy} + \frac{1}{2}yz_y - \frac{1}{3}z + h(s)z^{-2}\Big\}ds \\ &= 2\Big\{-2\int_L^{R(s)} z^{-3}z_y^2 dy + \frac{1}{2}\int_L^{R(s)} z^{-1}dy + \Big[-z^{-2}z_y\Big|_L^{R(s)} - \frac{1}{2}yz^{-1}\Big|_L^{R(s)}\Big] \\ &\qquad -\frac{1}{3}\int_L^{R(s)} z^{-1}dy + h(s)\int_L^{R(s)} z^{-4}dy\Big\} \\ &= \frac{1}{3}\int_L^{R(s)} z^{-1}dy + \Big(-e^{s/6} + Lz^{-1}(L) - 2z^{-2}z_y\Big|_L^{R(s)}\Big) \\ &\qquad -4\int_L^{R(s)} z^{-3}z_y^2 dy + 2h(s)\int_L^{R(s)} z^{-4}dy \\ &\triangleq M_{41} + M_{42} + M_{43} + M_{44}. \end{aligned}$$

Clearly,

$$|e^{-s/6}M_{42} + 1| \leqslant Ce^{-s/6}. \tag{4.13}$$

To estimate $M_{43}$, we use Lemma 2.4 to derive

$$|z_y(y,s)| \leqslant Ce^{-s/6}|ye^{-s/2}|^{-1/2} \leqslant Ce^{s/12}|y|^{-1/2} \leqslant Ce^{s/12} \quad \text{for } y \geqslant L.$$

It follows that

$$|M_{43}| \leqslant C\int_L^{R(s)} e^{s/12}z^{-2}z_y dy \leqslant Ce^{s/12}z^{-1}(L,s) \leqslant Ce^{s/12},$$

by using (3.18) twice, and so

$$e^{-s/6}|M_{43}| \leqslant Ce^{-s/12}. \tag{4.14}$$

To estimate $M_{44}$, by (2.1), we have

$$z^{-4}(y,s) \leqslant c^{-4}e^{(4/\beta-4/3)s}|y|^{-8/\beta}.$$

Hence, if we choose $\beta$ so that $0 < \gamma_2 \leqslant \frac{1}{6} - \frac{4}{\beta} + \frac{4}{3}$, we get

$$e^{-s/6}|M_{44}| \leqslant Ce^{-s/6+(4/\beta-4/3)s}\int_L^\infty |y|^{-8/\beta}dy \leqslant Ce^{-\gamma_2 s}. \tag{4.15}$$

Combining all these estimates, we find

$$\begin{aligned}\Big|\frac{d}{ds}\Big(e^{-s/6}K_2\Big)\Big| \quad &\leqslant \quad Ce^{-\gamma_2 s}\Big(1+\int_{-R(s)}^{R(s)}\rho|z_s|dy\Big)\\ &\qquad +\Big|-\frac{1}{6}e^{-s/6}K_2+\frac{1}{3}e^{-s/6}\int_L^{R(s)}z^{-1}dy\Big|.\end{aligned}$$

The last term is estimated by

$$\begin{aligned}&\Big|-\frac{1}{6}e^{-s/6}K_2+\frac{1}{3}e^{-s/6}\int_L^{R(s)}z^{-1}dy\Big|\\ &= \frac{1}{6}e^{-s/6}\Big|K_2-\int_{|y|\geqslant L}z^{-1}dy\Big|\\ &\leqslant \frac{1}{6}e^{-s/6}\Big(\int_{|y|\leqslant L}z^{-1}dy+\int_{-R(s)}^{R(s)}\rho z^{-1}dy\Big)\\ &\leqslant Ce^{-\gamma_1 s},\end{aligned}$$

where (4.7) was used in the last inequality. Therefore, the lemma is proved. ∎

*4.3. The conclusion*

We shall now always assume $s \gg 1$ so that $0 < e^{-s/6}K_1 < \frac{1}{2}$ and $e^{-s/6}K_2 \geqslant \frac{1}{2}u_* > 0$. This is possible by using (4.6). Therefore, the series defined in (4.5) is uniformly convergent.

We now proceed to estimate

$$\lambda^{-1}h(s)\frac{dK_1}{ds} \quad = \quad (1+e^{-s/6}K_2)^{-2}\sum_{j=0}^{\infty}(-1)^j\Big(\frac{e^{-s/6}}{1+e^{-s/6}K_2}\Big)^j\frac{dK_1^{j+1}}{ds}.$$

Applying integration by parts, we have

$$\begin{aligned}\lambda^{-1}\int_{s_1}^{s_2}h(s)\frac{dK_1}{ds}ds \quad &= \quad \Big[\sum_{j=0}^{\infty}(-1)^j\frac{e^{-js/6}}{(1+e^{-s/6}K_2)^{j+2}}K_1^{j+1}\Big]\Big|_{s_1}^{s_2}\\ &\quad -\sum_{j=0}^{\infty}(-1)^j\int_{s_1}^{s_2}K_1^{j+1}(s)\frac{d}{ds}\Big\{\frac{e^{-js/6}}{(1+e^{-s/6}K_2)^{j+2}}\Big\}ds\\ &\triangleq \quad K_{21}+K_{22}.\end{aligned}$$

First, we write $K_{21}$ as

$$\Big[K_1 \frac{1}{(1+e^{-s/6}K_2)^2} \sum_{j=0}^{\infty} (-1)^j \left(\frac{e^{-s/6}K_1}{1+e^{-s/6}K_2}\right)^j\Big]\Big|_{s_1}^{s_2};$$

the quantity $\frac{1}{(1+e^{-s/6}K_2)^2} \sum_{j=0}^{\infty} (-1)^j \left(\frac{e^{-s/6}K_1}{1+e^{-s/6}K_2}\right)^j$ is uniformly bounded, by (4.6). Therefore,

$$|K_{21}| \leqslant C(|K_1(s_2)| + |K_1(s_1)|).$$

To estimate $K_{22}$, we compute

$$\begin{aligned} & \frac{d}{ds}\Big\{\frac{e^{-js/6}}{(1+e^{-s/6}K_2)^{j+2}}\Big\} \\ = & \quad -\frac{j}{6}\,\frac{e^{-js/6}}{(1+e^{-s/6}K_2)^{j+2}} - \frac{e^{-js/6}(j+2)}{(1+e^{-s/6}K_2)^{j+3}}\frac{d}{ds}\Big(e^{-s/6}K_2\Big), \end{aligned}$$

so that

$$\begin{aligned} K_{22} \quad = & \quad \sum_{j=1}^{\infty} (-1)^j \frac{j}{6} \int_{s_1}^{s_2} K_1^{j+1}(s)\, \frac{e^{-js/6}}{(1+e^{-s/6}K_2)^{j+2}} ds \\ & + \sum_{j=0}^{\infty} (-1)^j \int_{s_1}^{s_2} K_1^{j+1}(s) \frac{e^{-js/6}(j+2)}{(1+e^{-s/6}K_2)^{j+3}} \frac{d}{ds}\Big(e^{-s/6}K_2\Big) ds \\ \triangleq & \quad K_{221} + K_{222}. \end{aligned}$$

Clearly,

$$\begin{aligned} & \Big|\sum_{j=1}^{\infty}(-1)^j \frac{j}{6} K_1^{j+1}\, \frac{e^{-js/6}}{(1+e^{-s/6}K_2)^{j+2}}\Big| \\ = & \quad e^{-s/6}K_1^2 \Big|\sum_{j=1}^{\infty}(-1)^j \frac{j}{6}\, \frac{1}{(1+e^{-s/6}K_2)^3}\Big(\frac{e^{-s/6}K_1}{(1+e^{-s/6}K_2)}\Big)^{j-1}\Big| \\ \leqslant & \quad Ce^{-s/6}K_1^2, \end{aligned}$$

so that, by Lemma 4.2,

$$|K_{221}| \quad \leqslant \quad C \int_{s_1}^{s_2} e^{-s/6} K_1^2(s) ds \leqslant C.$$

Similarly,

$$\begin{aligned} & \Big|\sum_{j=0}^{\infty}(-1)^j K_1^{j+1} \frac{e^{-js/6}(j+2)}{(1+e^{-s/6}K_2)^{j+3}}\Big| \\ = & \quad \Big|\sum_{j=0}^{\infty}(-1)^j (j+2) \frac{K_1}{(1+e^{-s/6}K_2)^3}\Big(\frac{e^{-s/6}K_1}{(1+e^{-s/6}K_2)}\Big)^{j}\Big| \\ \leqslant & \quad CK_1, \end{aligned}$$

so that, by Lemma 4.3, for small $\varepsilon > 0$,

$$
\begin{aligned}
|K_{222}| &\leqslant C\int_{s_1}^{s_2} e^{-\gamma_2 s} K_1(s)\Big(1+\int_{-R(s)}^{R(s)} \rho|z_s|dy\Big)ds\\
&\leqslant C + C\int_{s_1}^{s_2} e^{-\gamma_2 s} K_1(s)\Big(\int_{-R(s)}^{R(s)} \rho|z_s|dy\Big)ds\\
&\leqslant C + \varepsilon\int_{s_1}^{s_2}\int_{-R(s)}^{R(s)} \rho|z_s|^2dy + C\varepsilon^{-1}\int_{s_1}^{s_2} e^{-2\gamma_2 s}K_1^2(s)ds\\
&\leqslant C\varepsilon^{-1} + \varepsilon\int_{s_1}^{s_2}\int_{-R(s)}^{R(s)} \rho|z_s|^2dy.
\end{aligned}
$$

Combining these estimates, we find

$$
|K_{22}| \leqslant C\varepsilon^{-1} + \varepsilon\int_{s_1}^{s_2}\int_{-R(s)}^{R(s)} \rho|z_s|^2dy.
$$

Since $\int_{s_1}^{s_1+1} K_1(s)ds < C$, we can choose $s_j \to \infty$ such that $K_1(s_j) < C$. Therefore, we have proved

**Lemma 4.4.** *There exists $s_j \to \infty$ such that*

$$
\lambda^{-1}\Big|\int_{s_1}^{s_j} h(s)\frac{dK_1}{ds}ds\Big| \leqslant C\varepsilon^{-1} + \varepsilon\int_{s_1}^{s_j}\int_{-R(s)}^{R(s)} \rho|z_s|^2dy \tag{4.16}
$$

*for some small positive constant $\varepsilon$.*

With Lemma 4.4, we are ready to verify that $E(s)$ is actually a Lyapunov function as follows. Indeed, by integrating (4.2) from $s_1$ to any $s_j$ and using (4.16), it follows that

$$
\begin{aligned}
\int_{s_1}^{s_j}\int_{-R(s)}^{R(s)} \rho(y)z_s^2dyds &= E(s_1) - E(s_j) + \int_{s_1}^{s_j} h(s)\frac{dK_1(s)}{ds}ds + \int_{s_1}^{s_j} J_1(s)ds\\
&\leqslant C\varepsilon^{-1} + \lambda\varepsilon\int_{s_1}^{s_j}\int_{-R(s)}^{R(s)} \rho(y)z_s^2dyds.
\end{aligned}
$$

Taking a fixed $\varepsilon \in (0, \lambda^{-1}/2)$, we obtain

**Lemma 4.5.** *It holds*

$$
\int_{s_1}^{\infty}\int_{-R(s)}^{R(s)} \rho(y)z_s^2dyds < \infty. \tag{4.17}
$$

Applying (4.17) to Lemma 3.3, we find that now that $z(y,s)$ is uniformly bounded from below by a positive constant, we can immediately apply parabolic estimates to derive uniform $C^{2+\alpha,1+\alpha/2}$ estimates. The standard method implies that $z(y,s)$ converges uniformly on any compact set to the constant $w_\infty$ (depending on the solution itself). This completes the proof of Theorem 2.

**Proof of Theorem 3:** Using the relationship (1.9) and apply Theorem 2, we conclude Theorem 3. ∎

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