STABILITY AND UNIQUENESS OF TRAVELING WAVES FOR A DISCRETE BISTABLE 3-SPECIES COMPETITION SYSTEM

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Abstract. We study the stability and uniqueness of nonzero speed traveling waves for a three-component lattice dynamical system. This system arises in the study of three species competition model in which there is no competition between the first and the third species. Under the bistable consideration, we first derive the strict monotonicity of nonzero speed traveling waves. Then some super-sub-solutions are constructed based on these strictly monotone traveling waves. Finally, utilizing the constructed super-sub-solutions, we prove the stability and uniqueness of nonzero speed traveling waves of this system.

1. Introduction

We consider the following lattice dynamical system:

$$u_j'(t) = d_1\mathcal{D}_2[u_j](t) + r_1u_j(t)[1-u_j(t)-b_2v_j(t)], \quad j\in\mathbb{Z},\ t\in\mathbb{R}, \tag{1.1}$$

$$v_j'(t) = d_2\mathcal{D}_2[v_j](t) + r_2v_j(t)[1-b_1u_j(t)-v_j(t)-b_3w_j(t)], \quad j\in\mathbb{Z},\ t\in\mathbb{R}, \tag{1.2}$$

$$w_j'(t) = d_3\mathcal{D}_2[w_j](t) + r_3w_j(t)[1-b_2v_j(t)-w_j(t)], \quad j\in\mathbb{Z},\ t\in\mathbb{R}. \tag{1.3}$$

where d_i, r_i, b_i, $i=1,2,3$, are positive constants and $\mathcal{D}_2[z_j](t) := z_{j+1}(t)+z_{j-1}(t)-2z_j(t)$. This system arises in the study of the competition between three species in which the habitat is divided into countably infinitely many niches. Here $u_j(t)$ (resp. $v_j(t)$, $w_j(t)$) denotes the population density of species u (resp. v, w) at niches j at time t. The constants r_i, $i=1,2,3$, are the intrinsic growth rates. The constants d_i, $i=1,2,3$, are diffusion coefficients. The nonlinearities in this system are taken so that the carrying capacity of each species is normalized to be 1 and there is no competition between species u and w.

In this paper, we shall focus on the following so-called bistable case. We assume that v is a strong competitor in the absence of species u (resp. w), namely,

$$b_2>1>b_1, \quad b_2>1>b_3. \tag{1.4}$$

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Hence species v is always the winner in the absence of species u (resp. w). However, we also assume that

$$b_1 + b_3 > 1. \tag{1.5}$$

Under the assumptions (1.4) and (1.5), the constant equilibria $(1, 0, 1)$ and $(0, 1, 0)$ are stable so that we have the bistable nonlinearity.

A natural ecological question is which species would win the competition. In order to answer this question, one usually study the existence of so-called traveling wave solutions connecting states $(1, 0, 1)$ and $(0, 1, 0)$ to be defined as follows. A traveling wave solution of (1.1)–(1.3) is a solution of the form

$$u_j(t) = \widehat{U}(j + ct), \quad v_j(t) = \widehat{V}(j + ct), \quad w_j(t) = \widehat{W}(j + ct)$$

for all $(j, t) \in \mathbb{Z} \times \mathbb{R}$ with some wave speed $c \in \mathbb{R}$ and wave profile $(\widehat{U}, \widehat{V}, \widehat{W})$ such that

$$\begin{cases} (u_j, v_j, w_j)(t) \to (1, 0, 1) \text{ as } j \to -\infty, \ t \in \mathbb{R}, \\ (u_j, v_j, w_j)(t) \to (0, 1, 0) \text{ as } j \to \infty, \ t \in \mathbb{R}. \end{cases} \tag{1.6}$$

We are interested in traveling wave solutions of (1.1)–(1.3) connecting $(1, 0, 1)$ and $(0, 1, 0)$ with values in $[0, 1]$. Therefore, finding a traveling wave solution of (1.1)–(1.3) is reduced to find a constant $c \in \mathbb{R}$ and functions $(\widehat{U}, \widehat{V}, \widehat{W}) \in [C^1(\mathbb{R})]^3$ such that $0 \le \widehat{U}, \widehat{V}, \widehat{W} \le 1$ on $\mathbb{R}$ and

$$c\widehat{U}'(\xi) = d_1 D_2[\widehat{U}](\xi) + r_1[\widehat{U}(1 - \widehat{U} - b_2\widehat{V})](\xi), \quad \xi \in \mathbb{R}, \tag{1.7}$$

$$c\widehat{V}'(\xi) = d_2 D_2[\widehat{V}](\xi) + r_2[\widehat{V}(1 - b_1\widehat{U} - \widehat{V} - b_3\widehat{W})](\xi), \quad \xi \in \mathbb{R}, \tag{1.8}$$

$$c\widehat{W}'(\xi) = d_3 D_3[\widehat{W}](\xi) + r_3[\widehat{W}(1 - b_2\widehat{V} - \widehat{W})](\xi), \quad \xi \in \mathbb{R}, \tag{1.9}$$

where $\xi := j + ct$ and $D_2[z](\xi) := z(\xi + 1) + z(\xi - 1) - 2z(\xi)$, with the asymptotic boundary conditions

$$(\widehat{U}, \widehat{V}, \widehat{W})(-\infty) = (1, 0, 1), \quad (\widehat{U}, \widehat{V}, \widehat{W})(+\infty) = (0, 1, 0). \tag{1.10}$$

The existence of traveling wave of (1.1)–(1.3) is derived in [12]. See also [10] for the corresponding monostable case. As for the existence, uniqueness and monotonicity of traveling wave of the 2-species discrete competition models, we refer the reader to [7, 25, 9, 8]. The monostable case is treated in [9] and the bistable case is treated in [7, 25].

In this paper, we are mainly interested in the stability and uniqueness of traveling wave solutions. The stability problem of traveling waves of parabolic equations and their discrete analog has been studied extensively in past years. See, for examples, the seminal work [5] on the scalar reaction–diffusion equation and the work [6, 16] for the two-component reaction–diffusion systems. In [6], the stability of traveling waves is proved by combining an idea of [5] and the invariance theory (cf. [24, 4]).

For the stability in discrete dynamics, we refer the reader to, for examples, [27, 2, 18, 3, 11, 13, 14] for one-component lattice dynamical systems. Both homogeneous and periodic heterogeneous media are treated. For the 2-component competition system with monostable nonlinearity, we refer the reader to [23]. The stability for the corresponding monostable case to system (1.1)–(1.3) is carried out recently, see [22]. However, little is done for the multi-component discrete dynamics with bistable nonlinearity. One of the main purposes of this work is to derive the stability of traveling waves for 3-component bistable lattice dynamical system (1.1)–(1.3). The same method here can be applied to obtain the stability of traveling waves to the 2-species strong competition (bistable) lattice dynamical system.

As for the uniqueness in bistable dynamics, it is expected that the wave speed is unique. This is true for the scalar reaction–diffusion equation and the two-component reaction–diffusion systems with bistable nonlinearity (cf. [15]). For the discrete dynamics, this question is more subtle. For small (discrete) diffusion coefficients, there is the so-called propagation failure discovered by Keener [17] for 1-component lattice dynamical system (see also [7] for the 2-component case). On the other hand, it is proved in [7] that nonzero wave speed is unique so that any wave cannot propagate in two different nonzero speeds. This is true also for the 3-component system (1.1)–(1.3). Moreover, we also derive the uniqueness (up to translations) of wave profiles to system (1.1)–(1.3). It is worth to remark that our method here is different from previous works mentioned above on uniqueness. The constructed super-sub-solutions play a key role in our analysis.

The rest of this paper is organized as follows. In Section 2, we derive the strict monotonicity of nonzero speed (monotone) traveling waves to system (1.1)–(1.3). In fact, we prove the strict monotonicity of any nonzero speed monotone traveling waves. Furthermore, we show that any nonzero speed traveling waves are strictly monotone under an extra condition (see (2.10) in Section 2) on the parameters. After proving the strict monotonicity of monotone waves with nonzero speed, in Section 3, we construct some suitable super-sub-solutions based on the traveling waves. Then, using these super-sub-solutions, we derive the stability of nonzero speed strictly monotone traveling waves in Section 4. Finally, applying the comparison theorem, in Section 4, we also prove that nonzero wave speed is unique and wave profile is unique up to translations.

2. Strict monotonicity of traveling waves

In [12], it is proved that there is a traveling wave solution $(c, \widehat{U}, \widehat{V}, \widehat{W})$ of (1.1)–(1.3) such that $\widehat{U}' \leq 0$, $\widehat{V}' \geq 0$ and $\widehat{W}' \leq 0$. In the sequel, we shall set $U := 1 - \widehat{U}$, $V := \widehat{V}$ and $W := 1 - \widehat{W}$. Then, by (1.7)–(1.9) and (1.10), $(c, U, V, W) \in \mathbb{R} \times [C^1(\mathbb{R})]^3$ is a solution of

the following problem:

$$cU'(\xi) = d_1 D_2[U](\xi) + f(U,V,W)(\xi), \quad \xi \in \mathbb{R}, \tag{2.1}$$

$$cV'(\xi) = d_2 D_2[V](\xi) + g(U,V,W)(\xi), \quad \xi \in \mathbb{R}, \tag{2.2}$$

$$cW'(\xi) = d_3 D_2[W](\xi) + h(U,V,W)(\xi), \quad \xi \in \mathbb{R}, \tag{2.3}$$

with $0 \le U, V, W \le 1$ in $\mathbb{R}$, $U'(\xi), V'(\xi), W'(\xi) \ge 0$ for all $\xi \in \mathbb{R}$ and

$$(U,V,W)(-\infty) = (0,0,0), \quad (U,V,W)(+\infty) = (1,1,1). \tag{2.4}$$

Here f, g, h are defined by

$$\begin{cases} f(u,v,w) := r_1(1-u)(b_2 v - u), \\ g(u,v,w) := r_2 v[1 - b_1(1-u) - v - b_3(1-w)], \\ h(u,v,w) := r_3(1-w)(b_2 v - w). \end{cases} \tag{2.5}$$

Note that (f,g,h) satisfies the following cooperative conditions

$$\frac{\partial f}{\partial v}, \frac{\partial f}{\partial w}, \frac{\partial g}{\partial u}, \frac{\partial g}{\partial w}, \frac{\partial h}{\partial u}, \frac{\partial h}{\partial v} \ge 0 \tag{2.6}$$

in the region $\{(u,v,w) \mid u \le 1, v \ge 0, w \le 1\}$. Furthermore,

$$(\bar{u}_j, \bar{v}_j, \bar{w}_j)(t) := (U,V,W)(j+ct), \quad j \in \mathbb{Z}, \quad t \in \mathbb{R},$$

is a traveling wave solution of

$$u_j'(t) = d_1 \mathcal{D}_2[u_j](t) + f(u_j(t), v_j(t), w_j(t)), \quad j \in \mathbb{Z},\ t \in \mathbb{R}, \tag{2.7}$$

$$v_j'(t) = d_2 \mathcal{D}_2[v_j](t) + g(u_j(t), v_j(t), w_j(t)), \quad j \in \mathbb{Z},\ t \in \mathbb{R}, \tag{2.8}$$

$$w_j'(t) = d_3 \mathcal{D}_2[w_j](t) + h(u_j(t), v_j(t), w_j(t)), \quad j \in \mathbb{Z},\ t \in \mathbb{R}. \tag{2.9}$$

We have the following strict monotonicity for monotone traveling waves with nonzero speed.

Proposition 2.1. *Let (c,U,V,W) be a monotone traveling wave of (2.7)–(2.9) connecting $(0,0,0)$ to $(1,1,1)$ with values in $[0,1]$. If $c \ne 0$, then $U'(\xi), V'(\xi), W'(\xi) > 0$ for all $\xi \in \mathbb{R}$.*

Proof. By (2.1)–(2.3) and (2.5), $U, V, W \in C^\infty(\mathbb{R})$. Suppose that $U'(\xi_0) = 0$ for some $\xi_0 \in \mathbb{R}$. Then U' attains its minimum at ξ_0, hence $U''(\xi_0) = 0$ and $D_2[U'](\xi_0) \ge 0$. Therefore, by differentiating (2.1) in ξ and using the cooperative conditions (2.6), we have

$$\begin{aligned} 0 = cU''(\xi_0) &= d_1 D_2[U'](\xi_0) + f_u(U(\xi_0), V(\xi_0), W(\xi_0))U'(\xi_0) \\ &\quad + f_v(U(\xi_0), V(\xi_0), W(\xi_0))V'(\xi_0) + f_w(U(\xi_0), V(\xi_0), W(\xi_0))W'(\xi_0) \\ &\ge d_1 D_2[U'](\xi_0). \end{aligned}$$

This implies that $U'(\xi_0 \pm 1) = 0$. Thus we inductively obtain $U'(\xi_0 + m) = 0$ for all $m \in \mathbb{Z}$ and hence if $c \ne 0$, there exists a sequence $\{t_m\}_{m \in \mathbb{Z}}$ with $t_m \to \pm\infty$ as $m \to \pm\infty$ such that $\bar{u}_0'(t_m) = 0$.

If $c > 0$, then $\bar{u}_j'(t), \bar{v}_j'(t), \bar{w}_j'(t) \geq 0$ and $\not\equiv 0$ for $j \in \mathbb{Z}$, $t \in \mathbb{R}$. Differentiating (2.7) with $j = 0$ in t, we have

$$\begin{aligned}\bar{u}_0''(t) &= d_1\mathcal{D}_2[\bar{u}_0'](t) + f_u(\bar{u}_0, \bar{v}_0, \bar{w}_0)\bar{u}_0'(t) + f_v(\bar{u}_0, \bar{v}_0, \bar{w}_0)\bar{v}_0'(t) + f_w(\bar{u}_0, \bar{v}_0, \bar{w}_0)\bar{w}_0'(t)\\ &\geq -(2d_1 + L_1)\bar{u}_0'(t),\end{aligned}$$

where $L_1 = \max_{0\leq u,v,w\leq 1} |f_u(u, v, w)| > 0$. Integrating the above inequality over an interval $[t, t_m]$ ($m \in \mathbb{N}$), we have $\bar{u}_0'(t_m) \geq \bar{u}_0'(t)\mathrm{e}^{-(2d_1+L_1)(t_m-t)}$ for all $t \leq t_m$. However, this contradicts the fact that $\bar{u}_0'(t_m) = 0$ for $m \in \mathbb{Z}$ and $\bar{u}_0' \not\equiv 0$.

Similarly, we can derive a contradiction for the case $c < 0$. Thus we obtain $U'(\xi) > 0$ for all $\xi \in \mathbb{R}$. We can also prove the strict monotonicity of V and W in the same manner. □

In fact, any wave profile $(\widehat{U}, \widehat{V}, \widehat{W})$ of a nonzero speed traveling wave of (1.1)–(1.3) is strictly monotone under the extra condition

$$d_1 = d_3, \quad r_1 \neq r_3. \tag{2.10}$$

More precisely, we can prove

Theorem 2.2. *Suppose that conditions* (1.4), (1.5) *and* (2.10) *hold. Then the wave profiles of any solution* $(c, \widehat{U}, \widehat{V}, \widehat{W})$ *with* $c \neq 0$ *to system* (1.7)–(1.9) *and* (1.10) *are strictly monotone, i.e.,* $\widehat{U}' < 0, \widehat{V}' > 0, \widehat{W}' < 0$ *in* $\mathbb{R}$.

Indeed, Theorem 2.2 can be proved exactly the same as that of [26, Theorem 1.2], since the monostable condition there plays no role in the proof. The proof there works well for the bistable condition here. For the reader's convenience, we shall briefly outline the proof as follows for the case $c > 0$. The case for $c < 0$ is similar.

To study the monotonicity of a function Z at $\pm\infty$ with $Z(\pm\infty) = 0$, we may study the limit of ratio $z(x) := Z'(x)/Z(x)$ as $x \to \pm\infty$. This can be done by applying a well-known theory from, e.g., [1, Theorem 4], namely, for a nonzero constant α, if a measurable function $z(x)$ satisfies

$$\alpha z(x) = \mathrm{e}^{\int_x^{x+1} z(s)\mathrm{d}s} + \mathrm{e}^{\int_x^{x-1} z(s)\mathrm{d}s} + B(x), \quad \forall\, x \in \mathbb{R}, \tag{2.11}$$

for some continuous function B with $B(\pm\infty) \in \mathbb{R}$, then the limits $\omega^{\pm} := \lim_{x\to\pm\infty} z(x)$ exist and are real roots to the equation

$$\alpha\omega = \mathrm{e}^{\omega} + \mathrm{e}^{-\omega} + B(\pm\infty).$$

Applying this method to (1.7)-(1.9), we immediately obtain

Proposition 2.3. *Let* $(c, \widehat{U}, \widehat{V}, \widehat{W})$ *be a solution to* (1.7)–(1.9) *and* (1.10) *with* $c > 0$. *Then the limits*

$$\lambda := \lim_{x\to+\infty} \frac{\widehat{U}'(x)}{\widehat{U}(x)}, \quad \mu := \lim_{x\to-\infty} \frac{\widehat{V}'(x)}{\widehat{V}(x)}, \quad \nu := \lim_{x\to+\infty} \frac{\widehat{W}'(x)}{\widehat{W}(x)} \tag{2.12}$$

exist such that λ *is the unique negative root of*

$$c\omega = d_1(\mathrm{e}^{\omega} + \mathrm{e}^{-\omega} - 2) + r_1(1 - b_2), \tag{2.13}$$

μ *is the unique positive root of*

$$c\omega = d_2(\mathrm{e}^{\omega} + \mathrm{e}^{-\omega} - 2) + r_2(1 - b_1 - b_3), \tag{2.14}$$

and ν *is the unique negative root of*

$$c\omega = d_3(\mathrm{e}^{\omega} + \mathrm{e}^{-\omega} - 2) + r_3(1 - b_2). \tag{2.15}$$

Indeed, set $\hat{u} := \widehat{U}'/\widehat{U}$. Dividing (1.7) by $\widehat{U}(\xi)$, we obtain

$$c\hat{u}(\xi) = d_1 \left\{ \frac{\widehat{U}(\xi+1)}{\widehat{U}(\xi)} + \frac{\widehat{U}(\xi-1)}{\widehat{U}(\xi)} - 2 \right\} + r_1[1 - \widehat{U}(\xi) - b_2\widehat{V}(\xi)], \quad \xi \in \mathbb{R}.$$

Noting that

$$\frac{\widehat{U}(\xi \pm 1)}{\widehat{U}(\xi)} = \exp\left\{ \int_{\xi}^{\xi\pm1} \hat{u}(s)\mathrm{d}s \right\},$$

we see that $\hat{u}$ satisfies (2.11) with

$$\alpha = c/d_1, \quad z = \hat{u}, \quad B(\xi) := r_1[1 - \widehat{U}(\xi) - b_2\widehat{V}(\xi)]/d_1 - 2.$$

Again, applying [1, Theorem 4] to (2.1)–(2.3), we obtain

Proposition 2.4. *Let* $(c, \widehat{U}, \widehat{V}, \widehat{W})$ *be a solution to* (1.7)–(1.9) *and* (1.10) *with* $c > 0$. *Then the limits*

$$\hat{\lambda} := \lim_{x\to-\infty} \frac{\widehat{U}'(x)}{1 - \widehat{U}(x)}, \quad \hat{\nu} := \lim_{x\to-\infty} \frac{\widehat{W}'(x)}{1 - \widehat{W}(x)} \tag{2.16}$$

exist such that either $-\hat{\lambda}$ *is the unique positive root of*

$$c\omega = d_1(\mathrm{e}^{\omega} + \mathrm{e}^{-\omega} - 2) - r_1, \tag{2.17}$$

or $-\hat{\lambda} = \mu$, *and, either* $-\hat{\nu}$ *is the unique positive root of*

$$c\omega = d_3(\mathrm{e}^{\omega} + \mathrm{e}^{-\omega} - 2) - r_3, \tag{2.18}$$

or $-\hat{\nu} = \mu$. *Here* μ *is defined in Proposition 2.3.*

Proof. Indeed, the proposition follows from the same proof as that for Lemmas 4.2 and 4.3 of [7], since (2.1) and (2.3) are exactly the same as equation (4.1) in [7]. See also the proof given in [26]. We therefore safely omit the details. □

We observe from (2.13) and (2.15) that $\lambda \neq \nu$ when (2.10) is imposed. Then we have the following result on the remaining limit of ratio. Since its proof is exactly the same as that in Section 2.2 of [26], we omit it here.

Proposition 2.5. *Let* $(c, \widehat{U}, \widehat{V}, \widehat{W})$ *be a solution to* (1.7)–(1.9) *and* (1.10) *with* $c > 0$. *Then, under condition* (2.10), *the limit*

$$\hat{\mu} := \lim_{x \to +\infty} \frac{\widehat{V}'(x)}{1 - \widehat{V}(x)} \tag{2.19}$$

exists such that either $-\hat{\mu}$ *is the unique negative root of*

$$c\omega = d_2(\mathrm{e}^{\omega} + \mathrm{e}^{-\omega} - 2) - r_2, \tag{2.20}$$

or $-\hat{\mu} = \max\{\lambda, \nu\}$, *where* λ *and* ν *are defined in Proposition 2.3.*

With Propositions 2.3–2.5 and applying a sliding method with the help of a strong comparison principle, we can prove Theorem 2.2 as that of [26, Theorem 1.2]. □

3. Construction of super-sub-solutions

Let $(\bar{u}_j, \bar{v}_j, \bar{w}_j)(t) = (U, V, W)(j + ct)$, $j \in \mathbb{Z}$, $t \in \mathbb{R}$, be a strictly monotone traveling wave of (2.7)–(2.9) with nonzero speed c, where (c, U, V, W) satisfies (2.1)–(2.3) and (2.4). In this section, we first construct super-sub-solutions of (2.7)–(2.9) based on the traveling wave. The strict monotonicity of (U, V, W) plays a crucial role for the construction.

As the standard notion, a function $(u_j^+, v_j^+, w_j^+)(t)$, $j \in \mathbb{Z}$, $t \geq 0$, is a super-solution of (2.7)–(2.9) if

$$\begin{cases} \dfrac{\mathrm{d}u_j^+}{\mathrm{d}t} \geq d_1 \mathcal{D}_2[u_j^+] + f(u_j^+, v_j^+, w_j^+), & j \in \mathbb{Z},\ t \geq 0, \\ \dfrac{\mathrm{d}v_j^+}{\mathrm{d}t} \geq d_2 \mathcal{D}_2[v_j^+] + g(u_j^+, v_j^+, w_j^+), & j \in \mathbb{Z},\ t \geq 0, \\ \dfrac{\mathrm{d}w_j^+}{\mathrm{d}t} \geq d_3 \mathcal{D}_2[w_j^+] + h(u_j^+, v_j^+, w_j^+), & j \in \mathbb{Z},\ t \geq 0. \end{cases} \tag{3.1}$$

A sub-solution is defined similarly as that in (3.1) but with the reverse inequalities.

Let J^- and J^+ be the Jacobian matrix of (f, g, h) evaluated at $(0, 0, 0)$ and $(1, 1, 1)$, respectively,

$$J^- = \begin{pmatrix} -r_1 & b_2 r_1 & 0 \\ 0 & -r_2(b_1 + b_3 - 1) & 0 \\ 0 & b_2 r_3 & -r_3 \end{pmatrix}, \quad J^+ = \begin{pmatrix} -r_1(b_2 - 1) & 0 & 0 \\ b_1 r_2 & -r_2 & b_3 r_2 \\ 0 & 0 & -r_3(b_2 - 1) \end{pmatrix}.$$

Since all eigenvalues of J^- and J^+ are negative under the assumptions (1.4) and (1.5), we can find matrices $K^\pm$ with $K^\pm_{ij} > J^\pm_{ij}$, $1 \le i, j \le 3$, such that all eigenvalues of $K^\pm$ have negative real parts. Here A_{ij} denotes the (i, j)-entry of a matrix A. Note that $K^\pm_{ij} > 0$ if $i \ne j$. Let

$$\lambda^\pm = \max\{\mathrm{Re}\,\lambda \mid \lambda \in \sigma(K^\pm)\} < 0, \tag{3.2}$$

where $\sigma(A)$ denotes the set of eigenvalues of a matrix A. Since all entries of $K^\pm + kI$ are positive for large $k > 0$, applying the Perron–Frobenius theorem [20, Corollary 3.2] to the irreducible matrices $K^\pm + kI$, we can find $\phi^\pm = (\phi^\pm_1, \phi^\pm_2, \phi^\pm_3)^\mathsf{T}$ with $\phi^\pm_i > 0$, $i = 1, 2, 3$, such that $(K^\pm + kI)\phi^\pm = (\lambda^\pm + k)\phi^\pm$. Here the symbol T denotes the transpose. Hence we have

$$K^\pm \phi^\pm = \lambda^\pm \phi^\pm. \tag{3.3}$$

Take $\nu > 0$ sufficiently small such that

$$J_{ij} < K^-_{ij} \quad \text{for } |u|, |v|, |w| < \nu, \tag{3.4}$$

$$J_{ij} < K^+_{ij} \quad \text{for } |u-1|, |v-1|, |w-1| < \nu, \tag{3.5}$$

where $J = \partial(f, g, h)/\partial(u, v, w)$ is the Jacobian matrix evaluated at (u, v, w). Then we can choose $R_1 = R_1(\nu) > 0$ such that

$$0 < U(\xi), V(\xi), W(\xi) < \nu/2 \quad \text{for } \xi < -R_1, \tag{3.6}$$

$$1 - \nu/2 < U(\xi), V(\xi), W(\xi) < 1 \qquad \text{for } \xi > R_1. \tag{3.7}$$

Let $\rho^\pm_i(x)$, $i = 1, 2, 3$, be smooth positive functions satisfying

$$(\rho^+_1(x), \rho^+_2(x), \rho^+_3(x))^\mathsf{T} \to p^-\phi^-,\ (\rho^-_1(x), \rho^-_2(x), \rho^-_3(x))^\mathsf{T} \to q^-\phi^-, \quad \text{as } x \to -\infty, \tag{3.8}$$

$$(\rho^+_1(x), \rho^+_2(x), \rho^+_3(x))^\mathsf{T} \to p^+\phi^+,\ (\rho^-_1(x), \rho^-_2(x), \rho^-_3(x))^\mathsf{T} \to q^+\phi^+, \quad \text{as } x \to +\infty, \tag{3.9}$$

in C^1-topology. Here the constants $p^\pm, q^\pm > 0$ are chosen such that $0 < p^-\phi^-_i < p^+\phi^+_i \le 1$ and $1 \ge q^-\phi^-_i > q^+\phi^+_i > 0$ for $i = 1, 2, 3$. Hence we may assume that ρ^+_1, ρ^+_2, ρ^+_3 are monotone increasing functions and ρ^-_1, ρ^-_2, ρ^-_3 are monotone decreasing functions satisfying $0 < \rho^\pm_i(x) \le 1$, $|(\rho^\pm_i)'(x)| \le 1$ for $x \in \mathbb{R}$ and $i = 1, 2, 3$.

Define

$$u^\pm_j(t) = U(j + ct + \xi^\pm \pm \sigma(1 - \mathrm{e}^{-\beta t})) \pm \sigma\delta \mathrm{e}^{-\beta t}\rho^\pm_1(j + ct + \xi^\pm), \tag{3.10}$$

$$v^\pm_j(t) = V(j + ct + \xi^\pm \pm \sigma(1 - \mathrm{e}^{-\beta t})) \pm \sigma\delta \mathrm{e}^{-\beta t}\rho^\pm_2(j + ct + \xi^\pm), \tag{3.11}$$

$$w^\pm_j(t) = W(j + ct + \xi^\pm \pm \sigma(1 - \mathrm{e}^{-\beta t})) \pm \sigma\delta \mathrm{e}^{-\beta t}\rho^\pm_3(j + ct + \xi^\pm), \tag{3.12}$$

where δ, β, σ are positive constants and $\xi^\pm \in \mathbb{R}$. Note that $u^\pm_j$, $v^\pm_j$, $w^\pm_j$ are nondecreasing in j. Then the following lemma holds.

Lemma 3.1. *Let $\beta = \min\{-\lambda^-, -\lambda^+\}/2 > 0$, where $\lambda^\pm$ are defined by (3.2). Then for any $\sigma > 0$, there exists some $\delta_0 = \delta_0(\sigma) > 0$ independent of $\xi^\pm$ such that for $\delta \in (0, \delta_0)$ the functions $(u_j^\pm, v_j^\pm, w_j^\pm)(t)$ satisfy*

$$\begin{cases} \dfrac{\mathrm{d}u_j^+}{\mathrm{d}t} > d_1\mathcal{D}_2[u_j^+] + f(u_j^+, v_j^+, w_j^+), & j \in \mathbb{Z},\ t \in \mathbb{R}, \\ \dfrac{\mathrm{d}v_j^+}{\mathrm{d}t} > d_2\mathcal{D}_2[v_j^+] + g(u_j^+, v_j^+, w_j^+), & j \in \mathbb{Z},\ t \in \mathbb{R}, \\ \dfrac{\mathrm{d}w_j^+}{\mathrm{d}t} > d_3\mathcal{D}_2[w_j^+] + h(u_j^+, v_j^+, w_j^+), & j \in \mathbb{Z},\ t \in \mathbb{R}, \end{cases} \tag{3.13}$$

and

$$\begin{cases} \dfrac{\mathrm{d}u_j^-}{\mathrm{d}t} < d_1\mathcal{D}_2[u_j^-] + f(u_j^-, v_j^-, w_j^-), & j \in \mathbb{Z},\ t \in \mathbb{R}, \\ \dfrac{\mathrm{d}v_j^-}{\mathrm{d}t} < d_2\mathcal{D}_2[v_j^-] + g(u_j^-, v_j^-, w_j^-), & j \in \mathbb{Z},\ t \in \mathbb{R}, \\ \dfrac{\mathrm{d}w_j^-}{\mathrm{d}t} < d_3\mathcal{D}_2[w_j^-] + h(u_j^-, v_j^-, w_j^-), & j \in \mathbb{Z},\ t \in \mathbb{R}, \end{cases}$$

respectively.

Proof. We only show the first inequality of (3.13), since the others can be treated in a similar manner.

Define

$$N_j(t) = \frac{\mathrm{d}u_j^+}{\mathrm{d}t}(t) - d_1\mathcal{D}_2[u_j^+](t) - f(u_j^+(t), v_j^+(t), w_j^+(t)).$$

Then, by (3.10)–(3.12), we have

$$\begin{aligned} N_j(t) = {} & U'(\xi)(c + \sigma\beta\mathrm{e}^{-\beta t}) + \sigma\delta\mathrm{e}^{-\beta t}[c(\rho_1^+)'(\eta) - \beta\rho_1^+(\eta)] \\ & - d_1\{D_2[U](\xi) + \sigma\delta\mathrm{e}^{-\beta t}D_2[\rho_1^+](\eta)\} \\ & - f(U(\xi) + \sigma\delta\mathrm{e}^{-\beta t}\rho_1^+(\eta), V(\xi) + \sigma\delta\mathrm{e}^{-\beta t}\rho_2^+(\eta), W(\xi) + \sigma\delta\mathrm{e}^{-\beta t}\rho_3^+(\eta)), \end{aligned}$$

where $\xi = j + ct + \xi^+ + \sigma(1 - \mathrm{e}^{-\beta t})$ and $\eta = j + ct + \xi^+$. In view of (2.1)–(2.3), we see that

$$N_j(t) = \sigma\mathrm{e}^{-\beta t}\{\beta U'(\xi) + \delta(I_1 - I_2)\}, \tag{3.14}$$

where

$$I_1 = c(\rho_1^+)'(\eta) - \beta\rho_1^+(\eta) - d_1D_2[\rho_1^+](\eta), \tag{3.15}$$

$$\begin{aligned} I_2 = {} & \rho_1^+(\eta)\int_0^1 f_u(u^\theta, v^\theta, w^\theta)\mathrm{d}\theta + \rho_2^+(\eta)\int_0^1 f_v(u^\theta, v^\theta, w^\theta)\mathrm{d}\theta \\ & + \rho_3^+(\eta)\int_0^1 f_w(u^\theta, v^\theta, w^\theta)\mathrm{d}\theta \end{aligned} \tag{3.16}$$

and

$$(u^\theta, v^\theta, w^\theta) = (U, V, W)(\xi) + \theta\sigma\delta\mathrm{e}^{-\beta t}(\rho_1^+, \rho_2^+, \rho_3^+)(\eta).$$

Define positive constants C and μ in the following way:

$$C = \beta + 2\max_{1\le i\le 3} d_i + \sum_{1\le i,j\le 3} |K_{ij}^+| + \sum_{1\le i,j\le 3} |K_{ij}^-|, \tag{3.17}$$

$$\mu = \beta \min\left\{p^- \min_{1\le i\le 3} \phi_i^-, q^+ \min_{1\le i\le 3} \phi_i^+\right\} \Big/ (|c| + C). \tag{3.18}$$

Then there exists some $R_2 = R_2(\mu) > 0$ satisfying

$$|\rho_i^+(x) - p^-\phi_i^-| < \mu,\ |\rho_i^-(x) - q^-\phi_i^-| < \mu \quad \text{for } x < -R_2,\ i = 1,2,3, \tag{3.19}$$

$$|\rho_i^+(x) - p^+\phi_i^+| < \mu,\ |\rho_i^-(x) - q^+\phi_i^+| < \mu \quad \text{for } x > R_2,\ i = 1,2,3, \tag{3.20}$$

$$|(\rho_i^\pm)'(x)| < \mu \quad \text{for } |x| > R_2,\ i = 1,2,3. \tag{3.21}$$

For any given $\sigma > 0$, we will show that $N_j(t) > 0$ for all $j \in \mathbb{Z}$, $t \in \mathbb{R}$, if δ is sufficiently small. We divide the proof into the following three cases:

(i) $\eta > R_0 := \max\{R_1(\nu), R_2(\mu) + 1\}$;
(ii) $\eta < -R_0$;
(iii) $-R_0 \le \eta \le R_0$.

We assume that $\delta_0 \le \nu/(2\sigma)$.

Case (i): Since $\eta > R_2(\mu) + 1$, (3.20) and (3.21) imply

$$I_1 \ge -\beta p^+\phi_1^+ - (\beta + |c| + 2d_1)\mu.$$

Then, since $\xi > R_1(\nu)$,

$$|u^\theta - 1| \le |u^\theta - U(\xi)| + |U(\xi) - 1| < \sigma\delta + \nu/2 \le \nu$$

for $\delta \in (0, \delta_0)$. Similarly, we have $|v^\theta - 1| < \nu$ and $|w^\theta - 1| < \nu$. Therefore, by (3.5), (3.20) and (3.3),

$$\begin{aligned} I_2 &\le \rho_1^+(\eta)K_{11}^+ + \rho_2^+(\eta)K_{12}^+ + \rho_3^+(\eta)K_{13}^+ \\ &\le p^+(K_{11}^+\phi_1^+ + K_{12}^+\phi_2^+ + K_{13}^+\phi_3^+) + \mu(|K_{11}^+| + K_{12}^+ + K_{13}^+) \\ &= \lambda^+ p^+\phi_1^+ + \mu(|K_{11}^+| + K_{12}^+ + K_{13}^+). \end{aligned}$$

Thus, using the definition of β, we obtain

$$\begin{aligned} N_j(t) &\ge \sigma \mathrm{e}^{-\beta t}[\beta U'(\xi) + \delta\{(-\beta - \lambda^+)p^+\phi_1^+ - (|c| + C)\mu\}] \\ &> \sigma\delta \mathrm{e}^{-\beta t}\{\beta p^+\phi_1^+ - (|c| + C)\mu\} \ge 0. \end{aligned}$$

Case (ii): We can show the inequality $N_j(t) > 0$ in a similar manner to Case (i).

Case (iii): Since $0 < \rho_1^+ \le 1$ and $|(\rho_1^+)'| \le 1$,

$$I_1 \ge -(|c| + \beta + 2d_1).$$

On the other hand, since

$$0 < U(\xi) \le u^\theta \le U(\xi) + \sigma\delta_0 < 1 + \nu/2 \quad \text{for } \delta \in (0, \delta_0)$$

and also $0 < v^\theta, w^\theta < 1 + \nu/2$, we see that

$$I_2 \le 3M, \quad M := \max_{1\le i,j\le 3}\left(\max_{-\nu/2\le u,v,w\le 1+\nu/2} |J_{ij}|\right),$$

where J_{ij} is the (i,j)-entry of the Jacobian matrix $\partial(f,g,h)/\partial(u,v,w)$ evaluated at (u,v,w). Note that the positive constant M depends only on ν.

Letting

$$\kappa = \min\left\{\min_{|\xi|\le R_0+\sigma} U'(\xi), \min_{|\xi|\le R_0+\sigma} V'(\xi), \min_{|\xi|\le R_0+\sigma} W'(\xi)\right\} > 0,$$

we see that

$$\begin{aligned} N_j(t) &\ge \sigma \mathrm{e}^{-\beta t}\{\beta U'(\xi) - \delta(\beta + |c| + 2d_1 + 3M)\} \\ &> \sigma \mathrm{e}^{-\beta t}\{\beta\kappa - \delta_0(|c| + C + 3M)\}, \end{aligned}$$

hence we have $N_j(t) > 0$ if

$$\delta_0 \le \frac{\beta\kappa}{|c| + C + 3M}.$$

The proof is completed. □

Hereafter we write

$$\begin{aligned} (u_1, v_1, w_1) &\le (u_2, v_2, w_2) \quad \text{if } u_1 \le u_2, v_1 \le v_2, w_1 \le w_2, \\ (u_1, v_1, w_1) &< (u_2, v_2, w_2) \quad \text{if } u_1 < u_2, v_1 < v_2, w_1 < w_2. \end{aligned}$$

Let $(u_j^1, v_j^1, w_j^1)(t)$ and $(u_j^2, v_j^2, w_j^2)(t)$, $j \in \mathbb{Z}$, $t \ge 0$, be a subsolution and a supersolution of (2.7)–(2.9), respectively, and assume that $(0,0,0) \le (u_j^k(t), v_j^k(t), w_j^k(t)) \le (1,1,1)$ for all $j \in \mathbb{Z}, t \ge 0, k = 1, 2$.

By the cooperative property (2.6), if $(u_j^1, v_j^1, w_j^1)(0) \le (u_j^2, v_j^2, w_j^2)(0)$ for $j \in \mathbb{Z}$, then we have $(u_j^1, v_j^1, w_j^1)(t) \le (u_j^2, v_j^2, w_j^2)(t)$ for all $j \in \mathbb{Z}, t \ge 0$. In fact, the above comparison principle for system (2.7)–(2.9) can be found in [21] for differential inequalities in ordered Banach space. Indeed, let the index set be $\mathbb{Z}$, define $z_k := u_j$ for $k = 3j+1$, $z_k := v_j$ for $k = 3j+2$ and $z_k := w_j$ for $k = 3j+3$, and let

$$F_k(\mathbf{z}) := \begin{cases} d_1 z_{k-3} - 2d_1 z_k + d_1 z_{k+3} + f(z_k, z_{k+1}, z_{k+2}), & k = 3j+1, \\ d_2 z_{k-3} - 2d_2 z_k + d_2 z_{k+3} + g(z_{k-1}, z_k, z_{k+1}), & k = 3j+2, \\ d_3 z_{k-3} - 2d_3 z_k + d_3 z_{k+3} + h(z_{k-2}, z_{k-1}, z_k), & k = 3j+3 \end{cases}$$

for $\mathbf{z} := (z_k)_{k\in\mathbb{Z}}$. Note that the cooperative conditions (2.6) are fulfilled only in the region $\{(u,v,w) \mid u \le 1, v \ge 0, w \le 1\}$. Then $(F_k)_{k\in\mathbb{Z}}$ is *quasimonotone increasing* in $\mathbf{z}$. Therefore, the theorem on p.100 of [21] implies the above comparison principle, since the condition on the modulus of continuity is clear.

Furthermore, the following strict comparison principle holds.

Lemma 3.2. *Let $(u_j^1, v_j^1, w_j^1)(t)$ and $(u_j^2, v_j^2, w_j^2)(t)$, $j \in \mathbb{Z}$, $t \geq 0$, be a subsolution and a supersolution of (2.7)–(2.9), respectively, such that $(0,0,0) \leq (u_j^k(t), v_j^k(t), w_j^k(t)) \leq (1,1,1)$ for all $j \in \mathbb{Z}, t \geq 0, k = 1,2$, and $(u_j^1, v_j^1, w_j^1)(0) \leq (u_j^2, v_j^2, w_j^2)(0)$ for $j \in \mathbb{Z}$. If, in addition, $u_j^1(0) \not\equiv 1, v_j^2(0) \not\equiv 0, w_j^1(0) \not\equiv 1$ and $(u_J^1, v_J^1, w_J^1)(0) \neq (u_J^2, v_J^2, w_J^2)(0)$ for some $J \in \mathbb{Z}$, then*

$$(u_j^1, v_j^1, w_j^1)(t) < (u_j^2, v_j^2, w_j^2)(t) \quad \textit{for all } j \in \mathbb{Z},\ t > 0.$$

Proof. Without loss of generality, we may assume $u_J^1(0) < u_J^2(0)$, since the other cases can be treated in the same manner. Set $(\mathcal{U}_j, \mathcal{V}_j, \mathcal{W}_j)(t) := (u_j^2 - u_j^1, v_j^2 - v_j^1, w_j^2 - w_j^1)(t)$ and choose a constant K with

$$K > \max\left\{2d_1 + \max_{u,v,w\in I}|f_u|, 2d_2 + \max_{u,v,w\in I}|g_v|, 2d_3 + \max_{u,v,w\in I}|h_w|\right\},$$

where $I = [0,1]$. Then

$$\begin{aligned}
&\mathrm{e}^{-Kt}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{U}_j(t)\mathrm{e}^{Kt}\right)\\
=\ & K\mathcal{U}_j(t) + d_1\mathcal{D}_2[\mathcal{U}_j](t) + f(u_j^2(t), v_j^2(t), w_j^2(t)) - f(u_j^1(t), v_j^1(t), w_j^1(t))\\
=\ & \left(K - 2d_1 + \int_0^1 f_u(su_j^2(t) + (1-s)u_j^1(t), v_j^2(t), w_j^2(t))\mathrm{d}s\right)\mathcal{U}_j(t) \qquad (3.22)\\
&+d_1\mathcal{U}_{j+1}(t) + d_1\mathcal{U}_{j-1}(t) + \int_0^1 f_v(u_j^1(t), sv_j^2(t) + (1-s)v_j^1(t), w_j^2(t))\mathrm{d}s\cdot\mathcal{V}_j(t)\\
&+\int_0^1 f_w(u_j^1(t), v_j^1(t), sw_j^2(t) + (1-s)w_j^1(t))\mathrm{d}s\cdot\mathcal{W}_j(t)
\end{aligned}$$

for $j \in \mathbb{Z}$ and $t \geq 0$.

Note that by the above comparison principle we have $(0,0,0) \leq (\mathcal{U}_j, \mathcal{V}_j, \mathcal{W}_j)(t)$ for all $t \geq 0$. Hence all terms on the right-hand side of (3.22) are nonnegative and so the assumption $\mathcal{U}_J(0) > 0$ implies $\mathcal{U}_J(t) > 0$ for all $t \geq 0$. Furthermore, in view of (3.22), we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{U}_j(t)\mathrm{e}^{Kt}\right) \geq d_1\mathcal{U}_{j\pm1}(t)\mathrm{e}^{Kt}, \quad j \in \mathbb{Z},\ t \geq 0.$$

This yields that

$$\mathcal{U}_{J\pm1}(t) \geq \mathrm{e}^{-Kt}\left(\mathcal{U}_{J\pm1}(0) + d_1\int_0^t \mathcal{U}_J(\tau)\mathrm{e}^{K\tau}\mathrm{d}\tau\right) > 0, \quad t > 0,$$

hence we inductively obtain $\mathcal{U}_{J\pm m}(t) > 0$ for all $t > 0$ and $m \in \mathbb{N}$.

For each $j \in \mathbb{Z}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{V}_j(t)\mathrm{e}^{Kt}\right) \geq \mathrm{e}^{Kt}\int_0^1 g_u(su_j^2(t) + (1-s)u_j^1(t), v_j^2(t), w_j^2(t))\mathrm{d}s\cdot\mathcal{U}_j(t), \quad j \in \mathbb{Z}, t > 0.$$

By assumption $v_j^2(0) \not\equiv 0$, it follows that $\mathcal{V}_j(t) > 0$ for all $t > 0$ for $j \in \mathbb{Z}$ satisfying $v_j^2(0) \neq 0$. Then, repeating the above argument implies $\mathcal{V}_j(t) > 0$ for all $j \in \mathbb{Z}$ and $t > 0$. Similarly, we have $\mathcal{W}_j(t) > 0$ for all $j \in \mathbb{Z}$ and $t > 0$. The proof is completed. □

Combining the above two lemmas, we have the following comparison principle for system (2.7)–(2.9).

Corollary 3.3. *Let $(u_j, v_j, w_j)(t)$, $j \in \mathbb{Z}$, $t \geq 0$, be a solution of (2.7)–(2.9) satisfying*

$$(0,0,0) \leq (u_j, v_j, w_j)(t) \leq (1,1,1) \quad \text{for } j \in \mathbb{Z},\ t \geq 0$$

and

$$(u_j^-, v_j^-, w_j^-)(0) < (u_j, v_j, w_j)(0) < (u_j^+, v_j^+, w_j^+)(0) \quad \text{for } j \in \mathbb{Z}$$

for some $\sigma > 0$, $\delta \in (0, \delta_0(\sigma))$ and $\xi_\pm \in \mathbb{R}$. Then

$$(u_j^-, v_j^-, w_j^-)(t) < (u_j, v_j, w_j)(t) < (u_j^+, v_j^+, w_j^+)(t) \quad \text{for } j \in \mathbb{Z},\ t \geq 0. \tag{3.23}$$

Proof. For $j \in \mathbb{Z}$, $t \geq 0$, set

$$(u_j^1, v_j^1, w_j^1)(t) = (\max\{u_j^-(t), 0\}, \max\{v_j^-(t), 0\}, \max\{w_j^-(t), 0\}),$$
$$(u_j^2, v_j^2, w_j^2)(t) = (\min\{u_j^+(t), 1\}, \min\{v_j^+(t), 1\}, \min\{w_j^+(t), 1\}).$$

Then (u_j^1, v_j^1, w_j^1) is a subsolution and (u_j^2, v_j^2, w_j^2) is a supersolution of (2.7)–(2.9). Since (u_j, v_j, w_j) is both a supersolution and a subsolution of (2.7)–(2.9), the inequality (3.23) follows from the strict comparison principle for (2.7)–(2.9). □

4. Stability and uniqueness of monotone traveling waves

Let $I = [0,1]$ and let $I^{\mathbb{Z}} := \{\mathbf{u} = (u_j)_{j\in\mathbb{Z}} \mid u_j \in I,\ j \in \mathbb{Z}\}$ be the set of functions on the lattice $\mathbb{Z}$ with values in I and define $\|\mathbf{u}\| := \sup_{j\in\mathbb{Z}} |u_j|$ for $\mathbf{u} = (u_j)_{j\in\mathbb{Z}} \in I^{\mathbb{Z}}$. Then $X = (I^{\mathbb{Z}})^3$ is a closed subset of the Banach space $(\ell^\infty(\mathbb{Z}))^3$ endowed with the norm $\|(\mathbf{u}, \mathbf{v}, \mathbf{w})\|_\infty := \max\{\|\mathbf{u}\|, \|\mathbf{v}\|, \|\mathbf{w}\|\}$.

Theorem 4.1. *Let $(\bar{\mathbf{u}}(t), \bar{\mathbf{v}}(t), \bar{\mathbf{w}}(t))$,*

$$\bar{\mathbf{u}}(t) := (U(j+ct))_{j\in\mathbb{Z}}, \quad \bar{\mathbf{v}}(t) := (V(j+ct))_{j\in\mathbb{Z}}, \quad \bar{\mathbf{w}}(t) := (W(j+ct))_{j\in\mathbb{Z}},$$

be a monotone traveling wave of (2.7)–(2.9) with speed $c \neq 0$ and let $(\mathbf{u}(t), \mathbf{v}(t), \mathbf{w}(t))$,

$$\mathbf{u}(t) := (u_j(t))_{j\in\mathbb{Z}}, \quad \mathbf{v}(t) := (v_j(t))_{j\in\mathbb{Z}}, \quad \mathbf{w}(t)) := (w_j(t)))_{j\in\mathbb{Z}},$$

be a solution of (2.7)–(2.9) with $(\mathbf{u}(0), \mathbf{v}(0), \mathbf{w}(0)) = (\mathbf{u}_0, \mathbf{v}_0, \mathbf{w}_0) \in X$.
Then $(\bar{\mathbf{u}}(t), \bar{\mathbf{v}}(t), \bar{\mathbf{w}}(t))$ is stable in the sense of Lyapunov, namely, for any $\varepsilon > 0$, there exists some constant $\gamma > 0$ such that

$$\|(\mathbf{u}(t), \mathbf{v}(t), \mathbf{w}(t)) - (\bar{\mathbf{u}}(t), \bar{\mathbf{v}}(t), \bar{\mathbf{w}}(t))\|_\infty < \varepsilon \quad \text{for all } t \geq 0,$$

if $(\mathbf{u}_0, \mathbf{v}_0, \mathbf{w}_0)$ satisfies

$$(0,0,0) \leq (u_{0j}, v_{0j}, w_{0j}) \leq (1,1,1) \quad \text{for all } j \in \mathbb{Z}, \tag{4.1}$$

where $\mathbf{u}_0 := (u_{0j})_{j\in\mathbb{Z}}, \mathbf{v}_0 := (v_{0j})_{j\in\mathbb{Z}}, \mathbf{w}_0 := (w_{0j})_{j\in\mathbb{Z}}$, *and*

$$\|(\mathbf{u}_0, \mathbf{v}_0, \mathbf{w}_0) - (\bar{\mathbf{u}}(0), \bar{\mathbf{v}}(0), \bar{\mathbf{w}}(0))\|_\infty < \gamma. \tag{4.2}$$

Proof. Let $\varepsilon > 0$ be a given small constant and let β be the constant in Lemma 3.1. Set $\sigma = \varepsilon/(2G)$ with

$$G := \max\left\{\max_{\xi\in\mathbb{R}} U'(\xi), \max_{\xi\in\mathbb{R}} V'(\xi), \max_{\xi\in\mathbb{R}} W'(\xi)\right\}$$

and choose a positive constant $\delta < \min\{\delta_0(\sigma), G\}$, where $\delta_0(\sigma)$ is the constant in Lemma 3.1. Let $(\mathbf{u}^\pm(t), \mathbf{v}^\pm(t), \mathbf{w}^\pm(t))$ be the functions defined by (3.10)–(3.12) with the above choice of constants β, σ, δ and $\xi^\pm = 0$.

Choose a constant $\rho_0 > 0$ such that $\rho_0 \le \rho_i^\pm(\eta)$ for any $\eta \in \mathbb{R}$ and $i = 1, 2, 3$. Take $\gamma = \sigma\delta\rho_0$ and suppose that $(\mathbf{u}_0, \mathbf{v}_0, \mathbf{w}_0) \in X$ satisfies (4.1) and (4.2). Then the comparison theorem implies

$$(0, 0, 0) < (u_j, v_j, w_j)(t) < (1, 1, 1) \quad \text{for all } j \in \mathbb{Z},\ t > 0. \tag{4.3}$$

Furthermore,

$$(u_j^-, v_j^-, w_j^-)(0) < (u_{0j}, v_{0j}, w_{0j}) < (u_j^+, v_j^+, w_j^+)(0)$$

for all $j \in \mathbb{Z}$. In view of Corollary 3.3 and (4.3), we see that

$$(u_j^-, v_j^-, w_j^-)(t) < (u_j, v_j, w_j)(t) < (u_j^+, v_j^+, w_j^+)(t)$$

for all $j \in \mathbb{Z}$ and $t \ge 0$. Therefore, since $0 < \rho_i^\pm \le 1$ for $i = 1, 2, 3$, we have

$$\begin{aligned} u_j(t) - U(j + ct) &\ge U(j + ct - \sigma(1 - \mathrm{e}^{-\beta t})) - U(j + ct) - \sigma\delta, \\ &\ge -\sigma \max_{\xi\in\mathbb{R}} U'(\xi) - \sigma\delta, \\ u_j(t) - U(j + ct) &\le U(j + ct + \sigma(1 - \mathrm{e}^{-\beta t})) - U(j + ct) + \sigma\delta \\ &\le \sigma \max_{\xi\in\mathbb{R}} U'(\xi) + \sigma\delta, \end{aligned}$$

hence

$$\|\mathbf{u}(t) - \bar{\mathbf{u}}(t)\| \le \sigma G + \sigma\delta < 2\sigma G = \varepsilon.$$

Similarly, we obtain

$$\|\mathbf{v}(t) - \bar{\mathbf{v}}(t)\| < \varepsilon, \quad \|\mathbf{w}(t) - \bar{\mathbf{w}}(t)\| < \varepsilon.$$

The theorem is proved. □

We now state the uniqueness of traveling waves with nozero speed connecting two stable states as follows:

Theorem 4.2. *Let $(\bar{u}_j, \bar{v}_j, \bar{w}_j)(t) = (U, V, W)(j + ct)$, $j \in \mathbb{Z}$, $t \in \mathbb{R}$, be a monotone traveling wave of (2.7)–(2.9) with nonzero speed c. If there exists a traveling wave $(\widetilde{U}, \widetilde{V}, \widetilde{W})(j + \tilde{c}t)$ satisfying $0 \le \widetilde{U}, \widetilde{V}, \widetilde{W} \le 1$ and*

$$\lim_{\xi \to -\infty} (\widetilde{U}, \widetilde{V}, \widetilde{W})(\xi) = (0, 0, 0), \quad \lim_{\xi \to +\infty} (\widetilde{U}, \widetilde{V}, \widetilde{W})(\xi) = (1, 1, 1),$$

then $\tilde{c} = c$ and $(\widetilde{U}, \widetilde{V}, \widetilde{W})(\xi) = (U, V, W)(\xi + \xi_*)$ for some $\xi_* \in \mathbb{R}$.

Proof. Let $\rho_0 > 0$ be such that $\rho_0 \le \rho_i^{\pm}(\eta)$ for any $\eta \in \mathbb{R}$ and $i = 1, 2, 3$ and choose a constant $\delta \in (0, \delta_0(1))$, where $\delta_0(1)$ is the constant in Lemma 3.1 for $\sigma = 1$. Since $(U, V, W)(\xi)$ is strictly monotone increasing, there exist $\xi_\pm \in \mathbb{R}$ with $\xi_- < \xi_+$ such that

$$(U, V, W)(j + \xi_-) - \delta\rho_0(1, 1, 1) < (\widetilde{U}, \widetilde{V}, \widetilde{W})(j) < (U, V, W)(j + \xi_+) + \delta\rho_0(1, 1, 1)$$

for all $j \in \mathbb{Z}$. Then, by Corollary 3.3,

$$\begin{aligned}
&(U, V, W)(j + ct + \xi_- - (1 - \mathrm{e}^{-\beta t})) - \delta \mathrm{e}^{-\beta t}(\rho_1^-, \rho_2^-, \rho_3^-)(j + ct + \xi_-) \\
&< (\widetilde{U}, \widetilde{V}, \widetilde{W})(j + \tilde{c}t) \\
&< (U, V, W)(j + ct + \xi_+ + (1 - \mathrm{e}^{-\beta t})) + \delta \mathrm{e}^{-\beta t}(\rho_1^+, \rho_2^+, \rho_3^+)(j + ct + \xi_+)
\end{aligned}$$

for all $j \in \mathbb{Z}$ and $t \ge 0$. Taking $t_n = n/|c|$ and $j = j - \mathrm{sgn}(c)n$ for $n \in \mathbb{N}$, we have

$$\begin{aligned}
&(U, V, W)(j + \xi_- - (1 - \mathrm{e}^{-\beta t_n})) - \delta \mathrm{e}^{-\beta t_n}(\rho_1^-, \rho_2^-, \rho_3^-)(j + \xi^-) \\
&< (\widetilde{U}, \widetilde{V}, \widetilde{W})(j + n(\tilde{c} - c)/|c|) \\
&< (U, V, W)(j + \xi_+ + (1 - \mathrm{e}^{-\beta t_n})) + \delta \mathrm{e}^{-\beta t_n}(\rho_1^+, \rho_2^+, \rho_3^+)(j + \xi^+)
\end{aligned} \tag{4.4}$$

for all $j \in \mathbb{Z}$ and $n \in \mathbb{N}$.

Suppose $\tilde{c} \ne c$. Then, letting $n \to \infty$ in (4.4), we obtain

$$(U, V, W)(j + \xi_- - 1) \le (0, 0, 0) \text{ or } (1, 1, 1) \le (U, V, W)(j + \xi_+ + 1), \quad j \in \mathbb{Z},$$

which leads the contradiction to the fact that $(0, 0, 0) < (U, V, W)(\xi) < (1, 1, 1)$ for $\xi \in \mathbb{R}$.

In the case where $\tilde{c} = c$, letting $n \to \infty$ in (4.4), we obtain

$$(U, V, W)(j + \xi_- - 1) \le (\widetilde{U}, \widetilde{V}, \widetilde{W})(j) \le (U, V, W)(j + \xi_+ + 1), \quad j \in \mathbb{Z}.$$

Since $(U, V, W)(\xi)$ is strictly monotone increasing in ξ, the above inequalities yield that the constant

$$\xi_* := \sup\{\xi \in \mathbb{R} \mid (U, V, W)(j + \xi) \le (\widetilde{U}, \widetilde{V}, \widetilde{W})(j) \text{ for all } j \in \mathbb{Z}\}$$

is well-defined and bounded. Furthermore, for $j \in \mathbb{Z}$,

$$(U, V, W)(j + \xi_*) \le (\widetilde{U}, \widetilde{V}, \widetilde{W})(j). \tag{4.5}$$

First we show that equality holds for at least one component in (4.5) for some $j_0 \in \mathbb{Z}$. Suppose the contrary. Then $(U, V, W)(j + \xi_*) < (\widetilde{U}, \widetilde{V}, \widetilde{W})(j)$ for all $j \in \mathbb{Z}$ and hence the

comparison theorem yields $(U,V,W)(j+ct+\xi_*) < (\widetilde{U},\widetilde{V},\widetilde{W})(j+ct)$ for all $j\in\mathbb{Z}$ and $t\ge 0$. Since $c\neq 0$, we have

$$(U,V,W)(\xi+\xi_*) < (\widetilde{U},\widetilde{V},\widetilde{W})(\xi) \quad \text{for all } \xi\in\mathbb{R}. \tag{4.6}$$

Let $\nu>0$ be a small constant satisfying (3.4) and (3.5). Then there exist positive constants k_0 and ν_0 such that $\nu_0\le k_0\phi_i^{\pm}\le\nu$ for $i=1,2,3$, where $\phi^{\pm}=(\phi_1^{\pm},\phi_2^{\pm},\phi_3^{\pm})^{\mathsf{T}}$ are respectively the eigenfunction of $K^{\pm}$ defined in the previous section. Take $R_{\pm}\in\mathbb{R}$ such that

$$\begin{aligned}
0 < U(\xi+\xi_*+1), V(\xi+\xi_*+1), W(\xi+\xi_*+1) < \nu_0 &\quad \text{for } \xi\le R_-,\\
1-\nu_0 < U(\xi+\xi_*), V(\xi+\xi_*), W(\xi+\xi_*) < 1 &\quad \text{for } \xi\ge R_+,\\
0\le \widetilde{U}(\xi),\widetilde{V}(\xi),\widetilde{W}(\xi) < \nu_0 &\quad \text{for } \xi\le R_-,\\
1-\nu_0 < \widetilde{U}(\xi),\widetilde{V}(\xi),\widetilde{W}(\xi) \le 1 &\quad \text{for } \xi\ge R_+.
\end{aligned}$$

By (4.6), we can find $\varepsilon_0\in(0,1)$ satisfying

$$(U,V,W)(\xi+\xi_*+\varepsilon_0) < (\widetilde{U},\widetilde{V},\widetilde{W})(\xi) \quad \text{for } \xi\in[R_--1,R_++1].$$

For $t\ge 0$, we define $j_{\pm}(t)\in\mathbb{Z}$ by

$$j_+(t)=\min\{j\in\mathbb{Z}\mid j+ct\ge R_+\},\quad j_-(t)=\max\{j\in\mathbb{Z}\mid j+ct\le R_-\}.$$

Then, $(\hat{u}_j,\hat{v}_j,\hat{w}_j)(t)=(\widetilde{U},\widetilde{V},\widetilde{W})(j+ct)-(U,V,W)(j+ct+\xi_*+\varepsilon_0)$ satisfies

$$\begin{aligned}
\hat{u}_j(t),\hat{v}_j(t),\hat{w}_j(t) > 0 &\qquad \text{for} \quad j_-(t)\le j\le j_+(t),\\
-\nu_0 < \hat{u}_j(t),\hat{v}_j(t),\hat{w}_j(t) < \nu_0 &\qquad \text{for} \quad j\le j_-(t),\ j\ge j_+(t)
\end{aligned} \tag{4.7}$$

and

$$\hat{u}_j(t),\hat{v}_j(t),\hat{w}_j(t)\to 0 \quad \text{as } j\to\pm\infty.$$

Furthermore, $(\hat{u}_j,\hat{v}_j,\hat{w}_j)(t)$ solves the following linear system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\hat{u}_j\\ \hat{v}_j\\ \hat{w}_j\end{pmatrix}=\begin{pmatrix}d_1\mathcal{D}_2[\hat{u}_j]\\ d_2\mathcal{D}_2[\hat{v}_j]\\ d_3\mathcal{D}_2[\hat{w}_j]\end{pmatrix}+\widehat{J}_j(t)\begin{pmatrix}\hat{u}_j\\ \hat{v}_j\\ \hat{w}_j\end{pmatrix},\quad j\in\mathbb{Z}, \tag{4.8}$$

where

$$\widehat{J}_j(t)=(\widehat{J}_{mn}(j,t))_{1\le m,n\le 3},\quad \widehat{J}_{mn}(j,t)=\int_0^1 J_{mn}(j,t;s)\mathrm{d}s$$

and $J_{mn}(j,t;s)$ is the (m,n)-entry of the Jacobian matrix $\partial(f,g,h)/\partial(u,v,w)$ evaluated at $(u^s,v^s,w^s)=s(\widetilde{U},\widetilde{V},\widetilde{W})(j+ct)+(1-s)(U,V,W)(j+ct+\xi_*+\varepsilon_0)$. Since $0\le u^s,v^s,w^s\le 1$ for $s\in[0,1]$, the system (4.8) is of cooperation type and hence admits the maximum principle. Furthermore,

$$\begin{aligned}
0\le u^s,v^s,w^s < \nu_0 &\quad \text{for } j\le j_-(t),\\
1-\nu_0 < u^s,v^s,w^s \le 1 &\quad \text{for } j\ge j_+(t).
\end{aligned}$$

In view of (3.4) and (3.5), we see that for $m, n = 1, 2, 3$,

$$\widehat{J}_{mn}(j,t) < \begin{cases} K^-_{mn}, & j \le j_-(t), \\ K^+_{mn}, & j \ge j_+(t). \end{cases} \tag{4.9}$$

Next we will prove that

$$\hat{u}_j(t), \hat{v}_j(t), \hat{w}_j(t) \ge 0 \quad \text{for all } t \ge 0 \text{ and } j \le j_-(t),\ j \ge j_+(t). \tag{4.10}$$

Suppose that $\hat{u}_J(T) < 0$ for some $T \ge 0$ and $J \le j_-(T)$. Define $(p_j, q_j, r_j)^\mathsf{T}(t) = -k_0 \mathrm{e}^{\lambda^- t}\phi^-$ for $t \ge 0$ and $j \le j_-(t)$, where $\lambda^- < 0$ is the eigenvalue of K^- corresponding to ϕ^-. Note that $(p_j, q_j, r_j)^\mathsf{T}(t)$ is independent of $j \in \mathbb{Z}$ and hence satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} p_j \\ q_j \\ r_j \end{pmatrix} = \begin{pmatrix} d_1\mathcal{D}_2[p_j] \\ d_2\mathcal{D}_2[q_j] \\ d_3\mathcal{D}_2[r_j] \end{pmatrix} + K^-\begin{pmatrix} p_j \\ q_j \\ r_j \end{pmatrix}.$$

Since $p_j, q_j, r_j < 0$, it follows from (4.9) that $(p_j, q_j, r_j)^\mathsf{T}$ is a subsolution of (4.8) for $t \ge 0$, $j \le j_-(t)$. Furthermore,

$$\begin{aligned} (\hat{u}_j - p_j, \hat{v}_j - q_j, \hat{w}_j - r_j)(t) &> (0,0,0), \quad t \ge 0,\ j = j_-(t), \\ \lim_{j\to-\infty}(\hat{u}_j - p_j, \hat{v}_j - q_j, \hat{w}_j - r_j)(t) &> (0,0,0), \quad t \ge 0, \\ (\hat{u}_j - p_j, \hat{v}_j - q_j, \hat{w}_j - r_j)(0) &\ge (0,0,0), \quad j \le j_-(t). \end{aligned}$$

Therefore the maximum principle for (4.8) implies $(\hat{u}_j, \hat{v}_j, \hat{w}_j)(t) \ge (p_j, q_j, r_j)(t)$ for all $t \ge 0$, $j \le j_-(t)$. Set $J_l = J - \mathrm{sgn}(c)l$ for $l \in \mathbb{N}$. Then, $J_l \le j_-(T + l/|c|)$ and $\hat{u}_{J_l}(T + l/|c|) = \hat{u}(J)(T) < 0$ for all $l \in \mathbb{N}$. However, it contradicts the following fact:

$$\hat{u}_{J_l}(T + l/|c|) \ge p_{J_l}(T + l/|c|) = -k_0 \mathrm{e}^{\lambda^-(T+l/|c|)}\phi_1^- \to 0 \quad \text{as } l \to \infty.$$

This contradiction proves that $\hat{u}_j(t) \ge 0$ for all $t \ge 0$ and $j \le j_-(t)$. Similarly, we have $\hat{v}_j(t), \hat{w}_j(t) \ge 0$ for $t \ge 0$, $j \le j_-(t)$. We can also prove $\hat{u}_j(t), \hat{v}_j(t), \hat{w}_j(t) \ge 0$ for $t \ge 0$, $j \ge j_+(t)$ in the same manner. Thus (4.10) is proved.

In view of (4.7) and (4.10), we see that $(U, V, W)(j + ct + \xi_* + \varepsilon_0) \le (\widetilde{U}, \widetilde{V}, \widetilde{W})(j + ct)$ for all $t \ge 0$, $j \in \mathbb{Z}$, contradicting the definition of ξ_*. Therefore, equality holds for either of the three components in (4.5) for some $j_0 \in \mathbb{Z}$. We may assume $U(j_0 + \xi_*) = \widetilde{U}(j_0)$ since the other cases can be treated in a similar manner.

Suppose that equality does not hold in (4.5) for some $j \in \mathbb{Z}$. Then, by the strict comparison principle, we have

$$(U, V, W)(j + ct + \xi_*) < (\widetilde{U}, \widetilde{V}, \widetilde{W})(j + ct) \quad \text{for all } j \in \mathbb{Z},\ t > 0.$$

Taking $t = 1/|c| > 0$ and $j = j_0 - \mathrm{sgn}(c) \in \mathbb{Z}$, we see that $(U, V, W)(j_0 + \xi_*) < (\widetilde{U}, \widetilde{V}, \widetilde{W})(j_0)$, contradicting the assumption $U(j_0 + \xi_*) = \widetilde{U}(j_0)$. This contradiction proves that equality

holds in (4.5) for all $j \in \mathbb{Z}$. Hence $(U, V, W)(j + ct + \xi_*) = (\widetilde{U}, \widetilde{V}, \widetilde{W})(j + ct)$ for all $j \in \mathbb{Z}$ and $t \geq 0$. The theorem is proved. □

The stability and uniqueness of nonzero speed traveling waves of (1.1)–(1.3) immediately follows from Theorems 4.1 and 4.2.

Theorem 4.3. *Let* $(u_j, v_j, w_j)(t) = (\widehat{U}, \widehat{V}, \widehat{W})(j+ct)$, $j \in \mathbb{Z}$, $t \in \mathbb{R}$, *be a monotone traveling wave solution of* (1.1)–(1.3) *satisfying* (1.6) *and suppose that* $c \neq 0$. *Then the traveling wave is stable in the sense of Lyapunov. Furthermore, any traveling wave of* (1.1)–(1.3) *satisfying* (1.6) *is a time translation of* $(u_j, v_j, w_j)(t)$.

References

[1] X. Chen, S.-C. Fu, J.-S. Guo, *Uniqueness and asymptotics of traveling waves of monostable dynamics on lattices*, SIAM J. Math. Anal., 38 (2006), 233–258.

[2] X. Chen and J.-S. Guo, *Existence and asymptotic stability of travelling waves of discrete quasilinear monostable equations*, J. Differential Equations, 184 (2002), 549–569.

[3] X. Chen, J.-S. Guo and C.-C. Wu, *Traveling waves in discrete periodic media for bistable dynamics*, Arch. Ration. Mech. Anal., 189 (2008), 189–236.

[4] K. N. Chueh, C. C. Conley and J. A. Smoller, *Positively invariant regions for systems of nonlinear diffusion equations*, Indiana Univ. Math. J., 26 (1977), 373–392.

[5] P.C. Fife and J.B. McLeod, *The approach of solutions of nonlinear diffusion equations to travelling wave solutions*, Arch. Ration. Mech. Anal., 65 (1977), 335–361.

[6] R.A. Gardner, *Existence and stability of travelling wave solutions of competition models: a degree theoretic approach*, J. Differential Equations, 44 (1982), 343–364.

[7] J.-S. Guo and C.-H. Wu, *Wave propagation for a two-component lattice dynamical system arising in strong competition models*, J. Differential Equations, 250 (2011), 3504–3533.

[8] J.-S. Guo and C.-H. Wu, *Recent developments on wave propagation in 2-species competition systems*, Discrete Contin. Dyn. Syst. Ser. B, 17 (2012), 2713–2724.

[9] J.-S. Guo and C.-H. Wu, *Traveling wave front for a two-component lattice dynamical system arising in competition models*, J. Differential Equations, 252 (2012), 4357–4391.

[10] J.-S. Guo, Y. Wang, C.-H. Wu and C.-C. Wu, *The minimal speed of traveling wave solutions for a diffusive three species competition system*, Taiwanese J. Math., 19 (2015), 1805–1829.

[11] J.-S. Guo and C.-C. Wu, *Uniqueness and stability of traveling waves for periodic monostable lattice dynamical system*, J. Differential Equations, 246 (2009), 3818–3833.

[12] J.-S. Guo and C.-C. Wu, *The existence of traveling wave solutions for a bistable three-component lattice dynamical system*, J. Differential Equations, 260 (2016), 1445–1455.

[13] S. Guo and J. Zimmer, *Stability of traveling wavefronts in discrete reaction-diffusion equations with nonlocal delay effects*, Nonlinearity, 28 (2015), 463–492.

[14] C.-H. Hsu, J.-J. Lin and T.-S. Yang, *Stability for monostable wave fronts of delayed lattice differential equations*, J. Dynam. Differential Equations, 29 (2017), 323–342.

[15] Y. Kan-on, *Parameter dependence of propagation speed of travelling waves for competition-diffusion equations*, SIAM J. Math. Anal., 26 (1995), 340–363.

[16] Y. Kan-on and Q. Fang, *Stability of monotone travelling waves for competition-diffusion equations*, Japan J. Indust. Appl. Math., 13 (1996), 343–349.

[17] J.P. Keener, *Propagation and its failure in coupled systems of discrete excitable cells*, SIAM J. Appl. Math., 47 (1987), 556–572.

[18] S. Ma and X. Zou, *Existence, uniqueness and stability of traveling waves in a discrete reaction-diffusion monostable equation with delay*, J. Differential Equations, 217 (2005), 54–87.

[19] T. Ogiwara and H. Matano, *Monotonicity and convergence results in order-preserving systems in the presence of symmetry*, Discrete Contin. Dyn. Syst., 5 (1999), 1–34.

[20] H.J. Smith, *Monotone Dynamical Systems, An Introduction to the Theory of Competitive and Cooperative Systems*, Mathematical Surveys and Monographs, **41**, American Mathematical Society, 1995.
[21] W. Walter, Differential and Integral Inequalities, Translated by L. Rosenblatt and L. Shampine, Springer-Verlag, New York, 1970.
[22] T. Su and G.-B. Zhang, *Stability of traveling wavefronts for a three-component Lotka-Volterra competition system on a lattice*, Electron. J. Differential Equations 2018, Paper No. 57, 16 pp.
[23] G. Tian, G.-B. Zhang, *Stabilty of traveling wavefronts for a discrete diffusive Lotka-Volterra competition system*, J. Math. Anal. Appl., 447 (2017), 222–242.
[24] H.F. Weinberger, *Invariant sets for weakly coupled parabolic and elliptic systems*, Rend. Mat., 8 (1975), 295–310.
[25] C.-C. Wu, *Existence of traveling wavefront for discrete bistable competition model*, Discrete Contin. Dyn. Syst. Ser. B, 16 (2011), 973–984.
[26] C.-C. Wu, *Monotonicity and uniqueness of wave profiles for a three components lattice dynamical system*, Discrete Contin. Dyn. Syst., 37 (2017), 2813–2827.
[27] B. Zinner, *Stability of traveling wavefronts for the discrete Nagumo equation*, SIAM J. Math. Anal., 22 (1991), 1016–1020.

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