Chapter Four

Utility

Preferences - A Reminder

- x $\succ$ y: x is preferred strictly to y.
- x $\sim$ y: x and y are equally preferred.
- x $\succsim$ y: x is preferred at least as much as is y.

Preferences - A Reminder

- Completeness: For any two bundles x and y it is always possible to state either that

  x $\succsim$ y

  or that

  y $\succsim$ x.

Preferences - A Reminder

- Reflexivity: Any bundle x is always at least as preferred as itself; *i.e.*

  x $\succsim$ x.

Preferences - A Reminder

- Transitivity: If
  x is at least as preferred as y, and
  y is at least as preferred as z, then
  x is at least as preferred as z; *i.e.*

  $x \succsim y$ and $y \succsim z \Rightarrow x \succsim z$.

Utility Functions

- A preference relation that is complete, reflexive, transitive and continuous can be represented by a continuous utility function.
- Continuity means that small changes to a consumption bundle cause only small changes to the preference level.

Utility Functions

- A utility function U(x) represents a preference relation $\succsim$ if and only if:

  $x' \succ x'' \Leftrightarrow U(x') > U(x'')$

  $x' \prec x'' \Leftrightarrow U(x') < U(x'')$

  $x' \sim x'' \Leftrightarrow U(x') = U(x'')$.

Utility Functions

- Utility is an ordinal (i.e. ordering) concept.
- *E.g.* if U(x) = 6 and U(y) = 2 then bundle x is strictly preferred to bundle y. But x is not preferred three times as much as is y.

Utility Functions & Indiff. Curves

- Consider the bundles (4,1), (2,3) and (2,2).
- Suppose (2,3) $\succ$ (4,1) ~ (2,2).
- Assign to these bundles any numbers that preserve the preference ordering; *e.g.* U(2,3) = 6 > U(4,1) = U(2,2) = 4.
- Call these numbers utility levels.

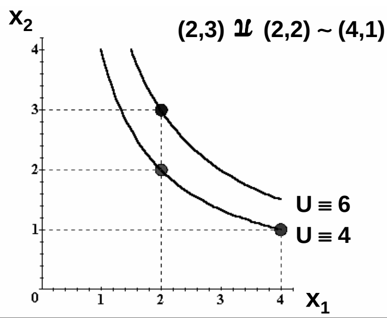
Utility Functions & Indiff. Curves

- An indifference curve contains equally preferred bundles.
- Equal preference $\Rightarrow$ same utility level.
- Therefore, all bundles in an indifference curve have the same utility level.

Utility Functions & Indiff. Curves

- So the bundles (4,1) and (2,2) are in the indiff. curve with utility level U $\equiv$ 4
- But the bundle (2,3) is in the indiff. curve with utility level U $\equiv$ 6.
- On an indifference curve diagram, this preference information looks as follows:

Utility Functions & Indiff. Curves

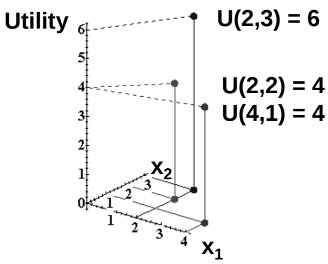


Utility Functions & Indiff. Curves

- Another way to visualize this same information is to plot the utility level on a vertical axis.

Utility Functions & Indiff. Curves

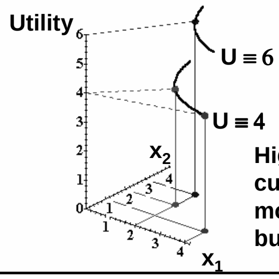
3D plot of consumption & utility levels for 3 bundles



Utility Functions & Indiff. Curves

- This 3D visualization of preferences can be made more informative by adding into it the two indifference curves.

Utility Functions & Indiff. Curves

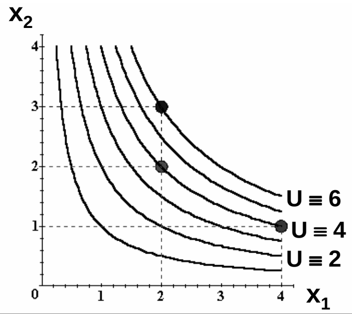


Higher indifference curves contain more preferred bundles.

Utility Functions & Indiff. Curves

- Comparing more bundles will create a larger collection of all indifference curves and a better description of the consumer's preferences.

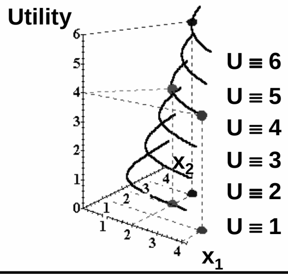
Utility Functions & Indiff. Curves



Utility Functions & Indiff. Curves

- As before, this can be visualized in 3D by plotting each indifference curve at the height of its utility index.

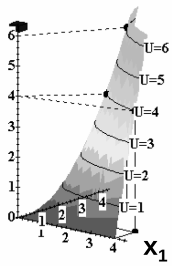
Utility Functions & Indiff. Curves



Utility Functions & Indiff. Curves

- Comparing all possible consumption bundles gives the complete collection of the consumer's indifference curves, each with its assigned utility level.
- This complete collection of indifference curves completely represents the consumer's preferences.

Utility Functions & Indiff. Curves



Utility Functions & Indiff. Curves

- The collection of all indifference curves for a given preference relation is an indifference map.
- An indifference map is equivalent to a utility function; each is the other.

Utility Functions

- There is no unique utility function representation of a preference relation.
- Suppose $U(x_1,x_2) = x_1x_2$ represents a preference relation.
- Again consider the bundles (4,1), (2,3) and (2,2).

Utility Functions

- $U(x_1,x_2) = x_1x_2$, so

  $U(2,3) = 6 > U(4,1) = U(2,2) = 4$;

  that is, $(2,3) \succ (4,1) \sim (2,2)$.

Utility Functions

- $U(x_1,x_2) = x_1x_2 \implies (2,3) \succ (4,1) \sim (2,2)$.
- Define $V = U^2$.
- Then $V(x_1,x_2) = x_1^2x_2^2$ and
  $V(2,3) = 36 > V(4,1) = V(2,2) = 16$
  so again
  $(2,3) \succ (4,1) \sim (2,2)$.
- V preserves the same order as U and so represents the same preferences.

Utility Functions

- $U(x_1,x_2) = x_1x_2 \implies (2,3) \succ (4,1) \sim (2,2)$.
- Define $W = 2U + 10$.
- Then $W(x_1,x_2) = 2x_1x_2+10$ so
  $W(2,3) = 22 > W(4,1) = W(2,2) = 18$. Again,
  $(2,3) \succ (4,1) \sim (2,2)$.
- W preserves the same order as U and V and so represents the same preferences.

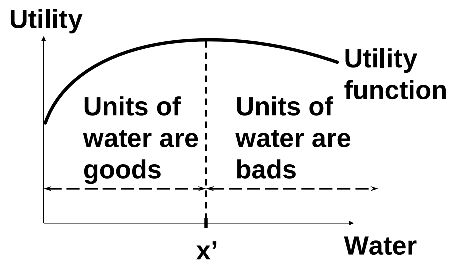
Utility Functions

- If
  - U is a utility function that represents a preference relation $\succsim$ and
  - f is a strictly increasing function,
- then $V = f(U)$ is also a utility function representing $\succsim$.

Goods, Bads and Neutrals

- A good is a commodity unit which increases utility (gives a more preferred bundle).
- A bad is a commodity unit which decreases utility (gives a less preferred bundle).
- A neutral is a commodity unit which does not change utility (gives an equally preferred bundle).

Goods, Bads and Neutrals



Around x' units, a little extra water is a neutral.

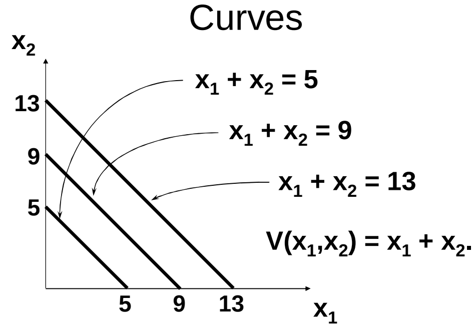
Some Other Utility Functions and Their Indifference Curves

- Instead of $U(x_1,x_2) = x_1x_2$ consider

$$V(x_1,x_2) = x_1 + x_2.$$

What do the indifference curves for this "perfect substitution" utility function look like?

Perfect Substitution Indifference Curves



All are linear and parallel.

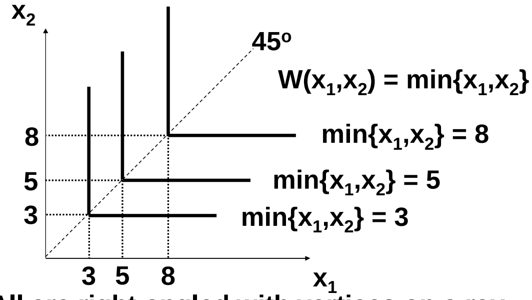
Some Other Utility Functions and Their Indifference Curves

- Instead of $U(x_1,x_2) = x_1x_2$ or $V(x_1,x_2) = x_1 + x_2$, consider

$$W(x_1,x_2) = \min\{x_1,x_2\}.$$

What do the indifference curves for this "perfect complementarity" utility function look like?

Perfect Complementarity Indifference Curves



All are right-angled with vertices on a ray from the origin.

Some Other Utility Functions and Their Indifference Curves

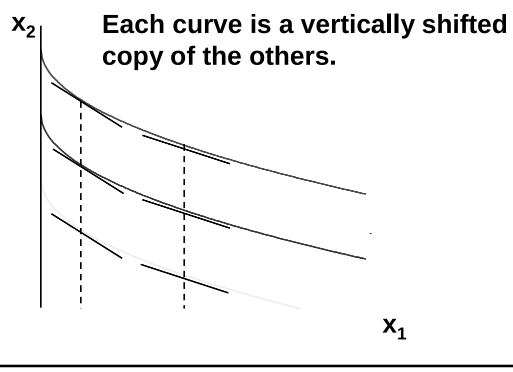
- A utility function of the form

$$U(x_1,x_2) = f(x_1) + x_2$$

is linear in just x_2 and is called quasi-linear.

- *E.g.* $U(x_1,x_2) = 2x_1^{1/2} + x_2$.

Quasi-linear Indifference Curves



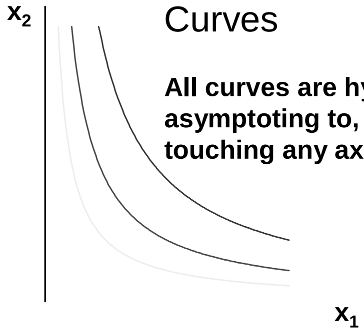
Some Other Utility Functions and Their Indifference Curves

- Any utility function of the form

  $U(x_1,x_2) = x_1^a x_2^b$

  with a > 0 and b > 0 is called a Cobb-Douglas utility function.
- *E.g.* $U(x_1,x_2) = x_1^{1/2} x_2^{1/2}$ (a = b = 1/2)
  $V(x_1,x_2) = x_1 x_2^3$ (a = 1, b = 3)

Cobb-Douglas Indifference Curves



All curves are hyperbolic, asymptoting to, but never touching any axis.

Marginal Utilities

- Marginal means "incremental".
- The marginal utility of commodity i is the rate-of-change of total utility as the quantity of commodity i consumed changes; *i.e.*

$$MU_i = \frac{\partial U}{\partial x_i}$$

Marginal Utilities

- *So,* if $U(x_1,x_2) = x_1^{1/2} x_2^2$ then

$$MU_1 = \frac{\partial U}{\partial x_1} = \frac{1}{2} x_1^{-1/2} x_2^2$$

$$MU_2 = \frac{\partial U}{\partial x_2} = 2 x_1^{1/2} x_2$$

Marginal Utility

- MU depends on the magnitude of utility, i.e it is cardinal.

Marginal Utilities and Marginal Rates-of-Substitution

- The general equation for an indifference curve is
  $U(x_1,x_2) \equiv k$, a constant.
  Totally differentiating this identity gives

$$\frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 = 0$$

Marginal Utilities and Marginal Rates-of-Substitution

$$\frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 = 0$$

rearranged is

$$\frac{\partial U}{\partial x_2} dx_2 = -\frac{\partial U}{\partial x_1} dx_1$$

Marginal Utilities and Marginal Rates-of-Substitution

And $\frac{\partial U}{\partial x_2} dx_2 = -\frac{\partial U}{\partial x_1} dx_1$

rearranged is

$$\frac{d\,x_2}{d\,x_1} = -\frac{\partial U / \partial x_1}{\partial U / \partial x_2}.$$

This is the MRS.

Marg. Utilities & Marg. Rates-of-Substitution; An example

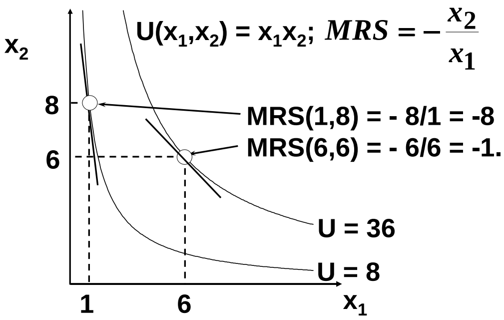
- Suppose $U(x_1,x_2) = x_1x_2$. Then

$$\frac{\partial U}{\partial x_1} = (1)(x_2) = x_2$$

$$\frac{\partial U}{\partial x_2} = (x_1)(1) = x_1$$

so $$MRS = \frac{dx_2}{dx_1} = -\frac{\partial U/\partial x_1}{\partial U/\partial x_2} = -\frac{x_2}{x_1}.$$

Marg. Utilities & Marg. Rates-of-Substitution; An example



Marg. Rates-of-Substitution for Quasi-linear Utility Functions

- A quasi-linear utility function is of the form $U(x_1,x_2) = f(x_1) + x_2$.

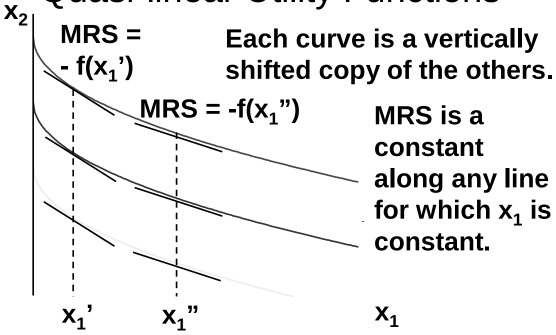
$$\frac{\partial U}{\partial x_1} = f'(x_1) \qquad \frac{\partial U}{\partial x_2} = 1$$

so $$MRS = \frac{dx_2}{dx_1} = -\frac{\partial U/\partial x_1}{\partial U/\partial x_2} = -f'(x_1).$$

Marg. Rates-of-Substitution for Quasi-linear Utility Functions

- MRS = $-f'(x_1)$ does not depend upon x_2 so the slope of indifference curves for a quasi-linear utility function is constant along any line for which x_1 is constant. What does that make the indifference map for a quasi-linear utility function look like?

Marg. Rates-of-Substitution for Quasi-linear Utility Functions



Monotonic Transformations & Marginal Rates-of-Substitution

- Applying a monotonic transformation to a utility function representing a preference relation simply creates another utility function representing the same preference relation.
- What happens to marginal rates-of-substitution when a monotonic transformation is applied?

Monotonic Transformations & Marginal Rates-of-Substitution

- For $U(x_1,x_2) = x_1x_2$ the MRS = $- x_2/x_1$.
- Create $V = U^2$; *i.e.* $V(x_1,x_2) = x_1^2x_2^2$. What is the MRS for V?

$$MRS = -\frac{\partial V / \partial x_1}{\partial V / \partial x_2} = -\frac{2x_1x_2^2}{2x_1^2x_2} = -\frac{x_2}{x_1}$$

which is the same as the MRS for U.

Monotonic Transformations & Marginal Rates-of-Substitution

- More generally, if V = f(U) where f is a strictly increasing function, then

$$MRS = -\frac{\partial V / \partial x_1}{\partial V / \partial x_2} = -\frac{f'(U) \times \partial U / \partial x_1}{f'(U) \times \partial U / \partial x_2}$$

$$= -\frac{\partial U / \partial x_1}{\partial U / \partial x_2}.$$

So MRS is unchanged by a positive monotonic transformation.

MU and MRS

- Diminishing MU cannot ensure diminishing MRS unless $U_{12} > 0$