CHAPTER 12

EXERCISE 12.2

1. Use the Lagrange-multiplier method to find the stationary values of z:
   (a) $z = xy$, subject to $x + 2y = 2$.
   (b) $z = x(y+4)$, subject to $x + y = 8$.
   (c) $z = x - 3y - xy$, subject to $x + y = 6$.
   (d) $z = 7 - y + x^2$, subject to $x + y = 0$

Ans:

(a) $Z = xy + \lambda(2 - x - 2y)$. The necessary condition is:

$$Z_\lambda = 2 - x - 2y = 0 \qquad Z_x = y - \lambda = 0 \qquad Z_y = x - 2\lambda = 0$$

Thus $\lambda^* = \frac{1}{2}$, $x^* = 1$, $y^* = \frac{1}{2}$-yielding $z^* = \frac{1}{2}$.

(b) $Z = xy + 4x + \lambda(8 - x - y)$. The necessary condition is:

$$Z_\lambda = 8 - x - y = 0 \qquad Z_x = y + 4 - \lambda = 0 \qquad Z_y = x - \lambda = 0$$

Thus $\lambda^* = 6$, $x^* = 6$, $y^* = 2$-yielding $z^* = 36$

(c) $Z = x - 3y - xy + \lambda(6 - x - y)$. The necessary condition is:

$$Z_\lambda = 6 - x - y = 0 \qquad Z_x = 1 - y - \lambda = 0 \qquad Z_y = -3 - x - \lambda = 0$$

Thus $\lambda^* = -4$, $x^* = 1$, $y^* = 5$-yielding $z^* = -19$

(d) $Z = 7 - y + x^2 + \lambda(-x - y)$. The necessary condition is:

$$Z_\lambda = -x - y = 0 \qquad Z_x = 2x - \lambda = 0 \qquad Z_y = -1 - \lambda = 0$$

Thus $\lambda^* = -1$, $x^* = -\frac{1}{2}$, $y^* = \frac{1}{2}$-yielding $z^* = 6\frac{3}{4}$

2. In Prob. 1, find whether a slight relaxation of the constraint will increase or decrease the optimal value of z. At what rate?

Ans:

(a) Increase; at the rate $\frac{dz^*}{dc} = \lambda^* = \frac{1}{2}$

(b) Increase; $\frac{dz^*}{dc} = 6$.

(c) Decrease; $\frac{dz^*}{dc} = -4$

(d) Decrease; $\frac{dz^*}{dc} = -1$

3. Write the Lagrangian function and the first-order condition for stationary values (without solving the equations) for each of the following:
   (a) $z = x + 2y + 3w + xy - yw$, subject to $x + y + 2w = 10$.
   (b) $z = x^2 + 2xy + yw^2$, subject to $2x + y + w^2 = 24$ and $x + w = 8$.

Ans:
(a) $Z = x + 2y + 3w + xy - yw + \lambda(10 - x - y - 2w)$. Hence:

$Z_\lambda = 10 - x - y - 2w = 0 \qquad Z_x = 1 + y - \lambda = 0$

$Z_y = 2 + x - w - \lambda = 0 \qquad Z_w = 3 - y - 2\lambda = 0$

(b) $Z = x^2 + 2xy + yw^2 + \lambda(24 - 2x - y - w^2) + v(8 - x - w)$. Thus:

$Z_\lambda = 24 - 2x - y - w^2 = 0 \qquad Z_v = 8 - x - w = 0$

$Z_x = 2x + 2y - 2\lambda - v = 0 \qquad Z_y = 2x - w^2 - \lambda = 0$

$Z_w = 2yw - 2\lambda w - v = 0$

4. If, instead of $g(x, y) = c$, the constraint is written in the form of $G(x, y) = 0$, how should the Lagrangian function and the first-order condition be modified as a consequence?

Ans: $Z = f(x, y) + \lambda[0 - G(x, y)] = f(x, y) - \lambda G(x, y)$. The first-order condition becomes: $Z_\lambda = -G(x, y) = 0 \quad Z_x = f_x - \lambda G_x = 0 \quad Z_y = f_y - \lambda G_y = 0$

5. In discussing the total-differential approach, it was pointed out that, given the constraint $g(x, y) = c$, we may deduce that $dg = 0$. By the same token, we can further deduce that $d^2g = d(dg) = d(0) = 0$. Yet, in our earlier discussion of the unconstrained extremum of a function $z = f(x, y)$, we had a situation where $dz = 0$ is accompanied by either a positive definite or a negative definite d^2z, rather than $d^2z = 0$. How would you account for this disparity of treatment in the two cases?

Ans: Since the constraint $g = c$ is to prevail at all times in this constrained optimization problem, the equation takes on the sense of an identity, and it follows that dg must be zero. Then it follows that $d^2 g$ must be zero, too. In contrast, the equation $dz = 0$ is in the nature of a first-order condition -- dz is not identically zero, but is being set equal to zero to locate the critical values of the choice variables. Thus $d^2 z$ does not have to be zero as a matter of course.

6. If the Lagrangian function is written as $Z = f(x, y) + \lambda[g(x, y) - c]$ rather than as in (12.7), can we still interpret the Lagrange multiplier as in (12.16)? Give the new interpretation, if any.

Ans: No, the sign of λ^* will be changed. The new λ^* is the negative of the old λ^*.

EXERCISE 12.3

1. Use the bordered Hessian to determine whether the stationary value of z obtained in each part of Exercise 12.2-1 is a maximum or a minimum.

Ans:

(a) Since $|\bar{H}| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix} = 4$, $z^* = \frac{1}{2}$ is a maximum.

(b) Since $|\bar{H}| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2$, $z^* = 36$ is a maximum

(c) Since $|\bar{H}| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{vmatrix} = -2$, $z^* = -19$ is a minimum

(d) Since $|\bar{H}| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -2$, $z^* = 6\frac{3}{4}$ is a minimum

2. In stating the second-order sufficient conditions for constrained maximum and minimum, we specified the algebraic signs of $|\bar{H}_2|$, $|\bar{H}_3|$, $|\bar{H}_4|$, etc., but

not of $|\overline{H}_1|$. Write out an appropriate expression for $|\overline{H}_1|$, and verify that it invariably takes the negative sign.

Ans: $|\overline{H}_1| = \begin{vmatrix} 0 & g_1 \\ g_1 & Z_{11} \end{vmatrix} = -g_1^2 < 0$

3. Recalling Property II of determinants (Sec. 5.3), show that:

   (a) By appropriately interchanging two rows and/or two columns of $|\overline{H}_2|$ and duly altering the sign of the determinant after each interchange, it can be transformed into

   (b) By a similar procedure, $|\overline{H}_3|$ can be transformed into

   What alternative way of "bordering" the principal minors of the Hessian do these results suggest?

Ans: The zero can be made that last (instead of the first) element in the principal diagonal, with g_1, g_2 and g_3 (in that order appearing in the last column and in the last row).

4. Write out the bordered Hessian for a constrained optimization problem with four choice variables and two constraints. Then state specifically the second-order sufficient condition for a maximum and for a minimum of z, respectively.

Ans: $$|\overline{H}| = \begin{vmatrix} 0 & 0 & g_1^1 & g_2^1 & g_3^1 & g_4^1 \\ 0 & 0 & g_1^2 & g_2^2 & g_3^2 & g_4^2 \\ g_1^1 & g_1^2 & Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ g_2^1 & g_2^2 & Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ g_3^1 & g_3^2 & Z_{31} & Z_{32} & Z_{33} & Z_{34} \\ g_4^1 & g_4^2 & Z_{41} & Z_{42} & Z_{43} & Z_{44} \end{vmatrix}$$

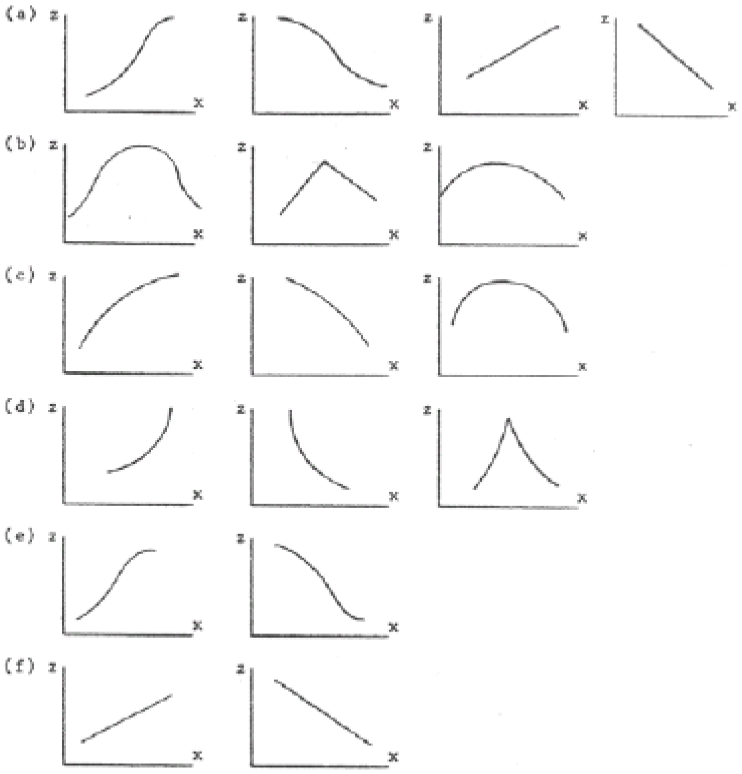
A sufficient condition for maximum z is $|\overline{H}_3| < 0$ and $|\overline{H}_4| = |\overline{H}| > 0$

A sufficient condition for minimum z is $|\overline{H}_3| > 0$ and $|\overline{H}| > 0$

EXERCISE 12.4

1. Draw a strictly quasiconcave curve $z = f(x)$ which is
   (a) also quasiconvex (d) not concave
   (b) not quasiconvex (e) neither concave nor convex
   (c) not convex (f) both concave and convex

Ans: Examples of acceptable curves are:



2. Are the following functions quasiconcave? Strictly so? First check graphically, and then algebraically by (12.20). Assume that $x \geq 0$.
   (a) $f(x) = a$ (b) $f(x) = a + bx$ $(b > 0)$ (c) $f(x) = a + cx^2$ $(c < 0)$

Ans:

(a) Quasiconcave, but not strictly so. This is because $f(v) = f(u) = a$, and thus $f[\theta u + (1-\theta)v] = a$, which is equal to (not greater than) $f(u)$.

(b) Quasiconcave, and strictly so. In the present case, $f(v) \geq f(u)$ means that $a + bv \geq a + bu$, or $v \geq u$. Moreover, to have u and v distinct, we must actually have $v > u$. Since

$$f[\theta u + (1-\theta)v] = a + b[\theta u + (1-\theta)v]$$
$$= a + b[\theta u + (1-\theta)v] + (bu - bu)$$
$$= a + bu + b(1-\theta)(v-u)$$

$= f(u)b(1-\theta)(v-u) = f(u)$ +some positive term

it follows that $f[\theta u + (1-\theta)v] > f(u)$. Hence $f(x) = a + bx$, $(b > 0)$, strictly quasiconcave.

(c) Quasiconcave, and strictly so. Here, $f(v) \geq f(u)$ means $a + cv^2 \geq a + cu^2$, or $v^2 \leq u^2$ (since $c < 0$). For nonnegative distinct values of u and v, this in turn means $v < u$. Now we have

$$f[\theta u + (1-\theta)v] = a + c[\theta u + (1-\theta)v]^2 + (cu^2 - cu^2)$$
$$= a + cu^2 + c\{[\theta u + (1-\theta)v]^2 - u^2\}$$

Using the identity $y^2 - x^2 \equiv (y+x)(y-x)$, we can rewrite the above expression as

$$a + cu^2 + c[\theta u + (1-\theta)v + u][\theta u + (1-\theta)v - u]$$
$$= f(u) + c[(1+\theta)u + (1-\theta)v][(1-\theta)(v-u)]$$
$$= f(u) \text{ + some positive term } > f(u)$$

Hence $f(x) = a + cx^2$, $(c < 0)$, is strictly quasiconcave.

3. (a) Let $z = f(x)$ plot as a negatively sloped curve shaped like the right half of a bell in the first quadrant, passing through the points (0,5), (2,4), (3,2), and (5,1). Let $z = g(x)$ plot as a positively sloped $45°$ line. Are f(x) and g(x) quasiconcave?

(b) Now plot the sum $f(x) + g(x)$. Is the sum function quasiconcave?

Ans: Both $f(x)$ and $g(x)$ are monotonic, and thus quasiconcave. However, $f(x) + g(x)$ displays both a hill and a valley. If we pick $k = 5\frac{1}{2}$, for instance, neither $S^{\geq}$ nor $S^{\leq}$ will be a convex set. Therefore $f(x) + g(x)$ is not quasiconcave.

4. By examining their graphs, and using (12.21), check whether the following functions are quasiconcave, quasiconvex, both, or neither:

(a) $f(x) = x^3 - 2x$ (b) $f(x_1, x_2) = 6x_1 - 9x_2$ (c) $f(x_1, x_2) = x_2 - \ln x_1$

Ans:

(a) This cubic function has a graph similar to Fig. 2.8c, with a hill in the second quadrant and valley in the fourth. If we pick $k = 0$, neither $S^{\geq}$ nor $S^{\leq}$ is a convex set. The function is neither quasiconcave nor quasiconvex.

(b) This function is linear, and hence both quasiconcave and quasiconvex.

(c) Setting $x_2 - \ln x_1 = k$, and solving for x_2, we get the isovalue equation $x_2 = \ln x_1 + k$. In the $x_1 x_2$ plane, this plots for each value of k as a log curve shifted upward vertically by the amount of k. The set

$S^{\le} = \{(x_1, x_2) \mid f(x_1, x_2) \le k\}$ – the set of points on or below the isovalue curve – is a convex set. Thus the function is quasiconvex. (but not quasiconcave).

5. (a) Verify that a cubic function $z = ax^3 + bx^2 + cx + d$ is in general neither quasiconcave nor quasiconvex.
   (b) Is it possible to impose restrictions on the parameters such that the function becomes both quasiconcave and quasiconvex for $x \ge 0$?

Ans:

(a) A cubic curve contains two bends, and would thus violate both parts of (12.21).

(b) From the discussion of the cubic total-cost function in Sec. 9.4, we know that if $a, c, d > 0$, $b < 0$, and $b^2 < 3ac$, then the cubic function will be upward-sloping for nonnegative x. Then, by (12.21), it is both quasiconcave and quasiconvex.

6. Use (12.22) to check $z = x^2 \ (x \ge 0)$ for quasiconcavity and quasiconvexity.

Ans: Let u and v be two values of x, and let $f(v) = v^2 \ge f(u) = u^2$, which implies $v \ge u$. Since $f'(x) = 2x$, we find that

$f'(u)(v-u) = 2u(v-u) \ge 0$

$f'(v)(v-u) = 2v(v-u) \ge 0$

Thus, by (12.22), the function is both quasiconcave and quasiconvex, confirming the conclusion in Example 1.

7. Show that $z = xy \ (x, y \ge 0)$ is not quasiconvex.

Ans: The set $S^{\le}$, involving the inequality $xy \le k$, consists of the points lying on or below a rectangular hyperbola – not a convex set. Hence the function is quasiconvex by (12.21). Alternatively, since $f_x = y$, $f_y = x$, $f_{xx} = 0$, $f_{xy} = 1$, and $f_{yy} = 0$, we have $|B_1| = -y^2 \le 0$ and $|B_2| = 2xy \ge 0$, which violates the necessary condition (12.25') for quasiconvexity.

8. Use bordered determinants to check the following functions for quasiconcavity and quasiconvexity:

(a) $z = -x^2 - y^2 \ (x, y > 0)$ (b) $z = -(x+1)^2 - (y+2)^2 \ (x, y > 0)$

Ans:

(a) Since $f_x = -2x$, $f_y = -2y$, $f_{xx} = -2$, $f_{yy} = -2$, $f_{xy} = 0$, we have

$|B_1| = -4x^2 < 0$ $|B_2| = 8(x^2 + y^2) > 0$

By (12.26), the function is quasiconcave.

(b) Since $f_x = -2(x+1)$, $f_y = -2(y+2)$, $f_{xx} = -2$, $f_{yy} = -2$, $f_{xy} = 0$, we have $|B_1| = -4(x+1)^2 < 0$ $|B_2| = 8(x+1)^2 + 8(y+2)^2 > 0$

By (12.26), the function is quasiconcave

EXERCISE 12.5

1. Given $U = (x+2)(y+1)$ and $P_x = 4$, $P_y = 6$, and $B = 130$:

(a) Write the Lagrangian function.
(b) Find the optimal levels of purchase x^* and y^*.
(c) Is the second-order sufficient condition for maximum satisfied?
(d) Does the answer in (b) give any comparative-static information?

Ans:

(a) $Z = (x+2)(y+1) + \lambda(130 - 4x - 6y)$

(b) The first-order condition requires that

$Z_\lambda = 130 - 4x - 6y = 0$, $Z_x = y + 1 - 4\lambda = 0$, $Z_y = x + 2 - 6\lambda = 0$

Thus we have $\lambda^* = 3$, $x^* = 16$, and $y^* = 11$.

(c) $|\overline{H}| = \begin{vmatrix} 0 & 4 & 6 \\ 4 & 0 & 1 \\ 6 & 1 & 0 \end{vmatrix} = 48 > 0$. Hence utility is maximized.

(d) No.

2. Assume that $U = (x+2)(y+1)$, but this time assign no specific numerical values to the price and income parameters.

(a) Write the Lagrangian function.

(b) Find x^*, y^*, and λ^* in terms of the parameters P_x, P_y, and B.

(c) Check the second-order sufficient condition for maximum.

(d) By setting $P_x = 4$, $P_y = 6$, and $B = 130$, check the validity of your answer to Prob. 1.

Ans:

(a) $Z = (x+2)(y+1) + \lambda(B - xP_x - yP_y)$

(b) As the necessary condition for extremum, we have

$$Z_\lambda = B - xP_x - yP_y = 0 \qquad \text{or} \qquad -P_x x - P_y y = -B$$

$$Z_x = y + 1 - \lambda P_x = 0 \qquad\qquad -P_x\lambda + y = -1$$

$$Z_y = x + 2 - \lambda P_y = 0 \qquad\qquad -P_y\lambda + x = -2$$

By Cramer's Rule, we can find that

$$\lambda^* = \frac{B + 2P_x + P_y}{2P_x P_y} \qquad x^* = \frac{B - 2P_x + P_y}{2P_x} \qquad y^* = \frac{B + 2P_x - P_y}{2P_y}$$

(c) $|\overline{H}| = \begin{vmatrix} 0 & P_x & P_y \\ P_x & 0 & 1 \\ P_y & 1 & 0 \end{vmatrix} = 2P_x P_y > 0$. Utility is maximized.

(d) When $P_x = 4$, $P_y = 6$, and $B = 130$, we get $\lambda^* = 3$, $x^* = 16$ and $y^* = 11$.

These check with preceding problem.

3. Can your solution (x^* and y^*) in Prob. 2 yield any comparative-static information? Find all the comparative-static derivatives you can, evaluate their signs, and interpret their economic meanings.

Ans: Yes. $\left(\frac{\partial x^*}{\partial B}\right) = \frac{1}{2P_x} > 0$, $\left(\frac{\partial x^*}{\partial P_x}\right) = -\frac{B + P_y}{2P_x^2} < 0$, $\left(\frac{\partial x^*}{\partial P_y}\right) = \frac{1}{2P_x} > 0$,

$\left(\frac{\partial y^*}{\partial B}\right) = \frac{1}{2P_y} > 0$, $\left(\frac{\partial y^*}{\partial P_x}\right) = \frac{1}{P_y} > 0$, $\left(\frac{\partial y^*}{\partial P_y}\right) = -\frac{B + 2P_x}{2P_y^2} < 0$.

An increase in income B raises the level of optimal purchases of x and y both; an increase in the price of one commodity reduces the optimal purchase of that commodity itself, but raises the optimal purchase of the other commodity.

4. From the utility function $U = (x+2)(y+1)$ and the constraint $xP_x + yP_y = B$ of Prob. 2, we have already found the U_{ij} and $|\overline{H}|$, as well as x^* and λ^*. Moreover, we recall that $|J| = |\overline{H}|$.
   (a) Substitute these into (12.39) and (12.40) to find $(\partial x^*/\partial B)$ and $(\partial y^*/\partial B)$.
   (b) Substitute into (12.42) and (12.43) to find $(\partial x^*/\partial P_x)$ and $(\partial y^*/\partial P_x)$.
   Do these results check with those obtained in Prob. 3?

Ans: We have $U_{xx^*} = U_{yy} = 0$, $U_{xy} = U_{yx^*} = 1$, $|J| = |\overline{H}| = 2P_xP_y$.

$x^* = \dfrac{B - 2P_x + P_y}{2P_x}$, and $\lambda^* = \dfrac{B + 2P_x + P_y}{2P_xP_y}$. Thus:

(a) $\left(\dfrac{\partial x^*}{\partial B}\right) = \dfrac{1}{2P_x}$, and $\left(\dfrac{\partial y^*}{\partial B}\right) = \dfrac{1}{2P_y}$.

(b) $\left(\dfrac{\partial x^*}{\partial P_x}\right) = -\dfrac{B + P_y}{2P_x^2}$, and $\left(\dfrac{\partial y^*}{\partial P_x}\right) = \dfrac{1}{P_y}$

These answers check with the preceding problem.

5. Comment on the validity of the statement: "If the derivative $(\partial x^*/\partial P_x)$ is negative, then x cannot possibly represent an inferior good."

Ans: A negative sign for that derivative can mean either that the income effect (T_1) and the substitution effect (T_2) in (12.33') are both negative (normal good), or that the income effect is positive (inferior good) but is overshadowed by the negative substitution effect. The statement is not valid.

6. When studying the effect of dP_x alone, the first equation in (12.37) reduces to $-P_x dx^* - P_y dy^* = x^* dP_x$, and when we compensate for the consumer's

effective income loss by dropping the term $x^* dP_x$, the equation becomes $-P_x dx^* - P_y dy^* = 0$. Show that this last result can be obtained alternatively from a compensation procedure whereby we try to keep the consumer's optimal utility level U^* (rather than effective income) unchanged, so that the term T_2 can alternatively be interpreted as $\left(\partial x^* / \partial P_x\right)_{U^* = \text{constant}}$. [Hint: Make use of (12.31").]

Ans: The optimal utility level can be expressed as $U^* = U^*(x^*, y^*)$. Thus $dU^* = U_x dx^* + U_y dy^*$, where U_x and U_y are evaluated at the optimum. When U^* is constant, we have $dU^* = 0$, or $U_x dx^* + U_y dy^* = 0$. From (12.42'), we have $\frac{U_x}{U_y} = \frac{P_x}{P_y}$ at the optimum. Thus we can also express $dU^* = 0$ by $P_x dx^* + P_y dy^* = 0$, or $-P_x dx^* - P_y dy^* = 0$.

7. (a) Does the assumption of diminishing marginal utility to goods x and y imply strictly convex indifference curves?
   (b) Does the assumption of strict convexity in the indifference curves imply diminishing marginal utility to goods x and y?

Ans:

(a) No; diminishing marginal utility means only that U_{xx} and U_{yy} are negative, but says nothing about U_{xy}. Therefore we cannot be sure that $|\overline{H}| > 0$ in (12.32) and $\frac{d^2y}{dx^2} > 0$ in (12.33').

(b) No; if $\frac{d^2y}{dx^2} > 0$, and hence $|\overline{H}| > 0$, nothing definite be said about the sign of U_{xx} and U_{yy}, because U_{xy} also appears in $|\overline{H}|$.

EXERCISE 12.6

1. Determine whether the following functions are homogeneous. If so, of what degree?

   (a) $f(x, y) = \sqrt{xy}$ (d) $f(x, y) = 2x + y + 3\sqrt{xy}$

   (b) $f(x, y) = (x^2 - y^2)^{1/2}$ (e) $f(x, y, w) = \frac{xy^2}{w} + 2xw$

   (c) $f(x, y) = x^3 - xy + y^3$ (f) $f(x, y, w) = x^4 - 5yw^3$

Ans:

(a) $\sqrt{(jx)(jy)} = j = \sqrt{xy}$; homogeneous of degree one.

(b) $\left[(jx)^2 - (jy)^2\right]^{1/2} = j(x^2 - y^2)^{1/2}$; homogeneous of degree one.

(c) Not homogeneous.

(d) $2jx + jy + 3\sqrt{(jx)(jy)} = j(2x + y + 3\sqrt{xy})$; homogeneous of degree one.

(e) $\frac{(jx)(jy)^2}{jw} + 2(jx)(jw) = j^2\left(\frac{xy^2}{w} + 2xw\right)$; homogeneous of degree two.

(f) $(jx)^4 - 5(jy)(jw)^3 = j^4(x^4 - 5yw^3)$; homogeneous of degree four.

2. Show that the function (12.45) can be expressed alternatively as $Q = K\psi\left(\frac{L}{K}\right)$ instead of $Q = L\phi\left(\frac{K}{L}\right)$.

Ans: Let $j = \frac{1}{k}$, then $\frac{Q}{K} = f\left(\frac{K}{K}, \frac{L}{K}\right) = f\left(\ell, \frac{L}{K}\right) = \psi\left(\frac{L}{K}\right)$. Thus $Q = K\psi\left(\frac{L}{K}\right)$.

(a) When $MPP_K = 0$, we have $L\frac{\partial Q}{\partial L} = Q$, or $\frac{\partial Q}{\partial L} = \frac{Q}{L}$, or $MPP_L = APP_L$.

(b) When $MPP_L = 0$, we have $K\frac{\partial Q}{\partial K} = Q$, or $\frac{\partial Q}{\partial K} = \frac{Q}{K}$, or $MPP_K = APP_K$.

3. Deduce from Euler's theorem that, with constant returns to scale:

   (a) When $MPP_K = 0$, APP_L is equal to MPP_L.

   (b) When $MPP_L = 0$, APP_K is equal to MPP_K.

Ans: Yes, they are true.

4. On the basis of (12.46) through (12.50), check whether the following are true under conditions of constant returns to scale:
   (a) An APP_L curve can be plotted against $k(=K/L)$ as the independent variable (on the horizontal axis).
   (b) MPP_K is measured by the slope of that APP_L curve.
   (c) APP_K is measured by the slope of the radius vector to the APP_L curve.
   (d) $MPP_L = APP_L - k(MPP_K) = APP_L - k$ (slope of APP_L)

Ans:
(a) $APP_L = \phi(k)$; hence APP_L indeed can be plotted against k.
(b) $MPP_K = \phi'(k) =$ slope of APP_L.
(c) $APP_K = \dfrac{\phi(k)}{k} = \dfrac{APP_L}{k} = \dfrac{\text{ordinate of point on the } APP_L \text{ curve}}{\text{abscissa of that point}}$

$= \text{slope of radius vector to the } APP_L \text{ curve}$
(d) $MPP_L = \phi(k) - k\phi'(k) = APP_L - k \cdot MPP_K$

5. Use (12.53) and (12.54) to verify that the relations described in Prob. 4b, c, and d are obeyed by the Cobb-Douglas production function.

Ans:
b. $APP_L = Ak^{\alpha}$, thus the slope of $APP_L = A\alpha k^{\alpha-1} = MPP_K$.
c. Slope of a radius vector $= \dfrac{Ak^{\alpha}}{k} = Ak^{\alpha-1} = APP_K$.
d. $APP_L - k \cdot MPP_K = Ak^{\alpha} - kA\alpha k^{\alpha-1} = Ak^{\alpha} - A\alpha k^{\alpha} = A(1-\alpha)k^{\alpha} = MPP_L$

6. Given the production function $Q = AK^{\alpha}L^{\beta}$, show that:
   (a) $\alpha + \beta > 1$ implies increasing returns to scale.
   (b) $\alpha + \beta < 1$ implies decreasing returns to scale.
   (c) α and β are, respectively, the partial elasticities of output with respect to the capital and labor inputs.

Ans:
(a) Since the function is homogeneous of degree $(\alpha + \beta)$, if $\alpha + \beta > 1$, the value of the function will increases more than j-fold when K and L are increase j-fold, implying increasing returns to scale.

(b) If $\alpha + \beta < 1$, the value of the function will increase less than j-fold when K and L are increased j-fold, implying decreasing returns to scale.

(c) Taking the natural log of both sides of the function, we have $\ln Q = \ln A + \alpha \ln K + \beta \ln L$. Thus $\varepsilon_{QK} = \frac{\partial(\ln Q)}{\partial(\ln K)} = \alpha$ and $\varepsilon_{QL} = \frac{\partial(\ln Q)}{\partial(\ln L)} = \beta$

7. Let output be a function of three inputs: $Q = AK^a L^b N^c$.
   (a) Is this function homogeneous? If so, of what degree?
   (b) Under what condition would there be constant returns to scale? Increasing returns to scale?
   (c) Find the share of product for input N, if it is paid by the amount of its marginal product.

Ans:

(a) $A(jK)^a (jL)^b (jN)^c = j^{a+b+c} \cdot AK^a L^b N^c = j^{a+b+c} Q$; homogeneous of degree $a + b + c$

(b) $a + b + c = 1$ implies constant returns to scale; $a + b + c > 1$ implies increasing returns to scale.

(c) Share for $N = \frac{N\left(\frac{\partial Q}{\partial N}\right)}{Q} = \frac{NAK^a L^b c N^{c-1}}{AK^a L^b N^c} = c$

8. Let the production function $Q = g(K, L)$ be homogeneous of degree 2.
   (a) Write an equation to express the second-degree homogeneity property of this function.
   (b) Find an expression for Q in terms of $\phi(k)$, in the vein of (12.45’).
   (c) Find the MPP_K function. Is MPP_K still a function of k alone, as in the linear-homogeneity case?
   (d) Is the MPP_K function homogeneous in K and L? If so, of what degree?

Ans:

(a) $j^2 Q = g(jK, jL)$

(b) Let $j = \frac{1}{L}$. Then the equation in (a) yield: $\frac{Q}{L^2} = g\left(\frac{K}{L}, 1\right) = \phi\left(\frac{K}{L}\right) = \phi(k)$.

This implies that $Q = L^2 \phi(k)$.

(c) $MPP_K = \frac{\partial Q}{\partial k} = L^2\phi'(k)\left(\frac{\partial k}{\partial K}\right) = L^2\phi'(k)\left(\frac{1}{L}\right) = L\phi'(k)$. Now MPP_K depends on L as well as k.

(d) If K and L are both increased j-fold in the MPP_K expression in (c), we get $(jL)\phi'\left(\frac{jK}{jL}\right) = jL\phi'\left(\frac{K}{L}\right) = jL\phi'(k) = j \cdot MPP_K$. Thus MPP_K is homogeneous of degree one in K and L.

EXERCISE 12.7

1. Suppose that the isoquants in Fig. 12.9b are derived from a particular homogeneous production function $Q = Q(a, b)$. Noting that $OE = EE' = E'EE''$, what must be the ratio between the output levels represented by the three isoquants if the function Q is homogeneous
   (a) of degree one? (b) of degree two?

Ans:

(a) Linear homogeneity implies that the output levels of the isoquants are in the ratio of 1:2:3 (from southwest to northeast).

(b) With second-degree homogeneity, the output levels are in the ratio of $1:2^2:3^2$, or 1:4:9.

2. For the generalized Cobb-Douglas case, if we plot the ratio b^*/a^* against the ratio P_a/P_b, what type of curve will result? Does this result depend on the assumption that $\alpha + \beta = 1$? Read the elasticity of substitution graphically from this curve.

Ans: Since $\left(\frac{b^*}{a^*}\right) = \left(\frac{\beta}{\alpha}\right)\left(\frac{P_a}{P_b}\right)$, it will plot as a straight line passing through the origin, with a (positive) slope equal to $\frac{\beta}{\alpha}$. This result does not depend on the assumption $\alpha + \beta = 1$. The elasticity of substitution is merely the elasticity of this line, which can be read (by the method of Fig. 8.2) to be unity at all points.

3. Is the CES production function characterized by diminishing returns to each input for all positive levels of input?

Ans: Yes, because Q_{LL} and Q_{KK} have both been found to be negative.

4. Show that, on an isoquant of the CES function, $d^2K/d^2L > 0$.

Ans: On the basis of (12.66), we have

$$\frac{d^2K}{dL^2} = \frac{d}{dL}\left[\frac{\delta-1}{\delta}\left(\frac{K}{L}\right)^{1+\rho}\right] = \frac{\delta-1}{\delta}(1+\rho)\left(\frac{K}{L}\right)^{\rho}\frac{d}{dL}\left(\frac{K}{L}\right)$$

$$= \frac{\delta-1}{\delta}(1+\rho)\left(\frac{K}{L}\right)^{\rho}\frac{1}{L^2}\left(L\frac{dK}{dL}-K\right) > 0 \qquad \text{(because } \frac{dK}{dL} < 0\text{)}$$

5. (a) For the CES function, if each factor of production is paid according to its marginal product, what is the ratio of labor's share of product to capital's share of product? Would a larger value of δ mean a larger relative share for capital?
(b) For the Cobb-Douglas function, is the ratio of labor's share to capital's share dependent on the K/L ratio? Does the same answer apply to the CES function?

Ans:

(a) $\dfrac{\text{Labor share}}{\text{Capital share}} = \dfrac{Lf_L}{Kf_K} = \dfrac{1-\delta}{\delta}\left(\dfrac{K}{L}\right)^{\rho}$. A larger ρ implies a larger capital share in relation to the labor share.

(b) No; no.

6. (a) The CES production function rules out $\rho = -1$. If $\rho = -1$, however, what would be the general shape of the isoquants for positive K and L?
(b) Is σ defined for $\rho = -1$? What is the limit of σ as $\rho \to -1$?
(c) Interpret economically the results for parts (a) and (b).

Ans:

(a) If $\rho = -1$, (12.66) yields $\dfrac{dK}{dL} = -\dfrac{(1-\delta)}{\delta} = \text{constant}$. The isoquants would be downward-sloping straight lines.

(b) By (12.68), σ is not defined for $\rho = -1$. But as $\rho \to -1$, $\sigma \to \infty$.

(c) Linear isoquants and infinite elasticity of substitution both imply that the two inputs are perfect substitutes.

7. Show that by writing the CES function as $Q = A\left[\delta K^{-\rho} + (1-\delta)L^{-\rho}\right]^{-r/\rho}$, where $r > 0$ is a new parameter, we can introduce increasing returns to scale and decreasing returns to scale.

Ans: If both K and L are changed j-fold, output will change from Q to:

$$A\left[\delta(jk)^{-\rho} + (1-\delta)(jL)^{-\rho}\right]^{-\frac{r}{\rho}} = A\left\{j^{-\rho}\left[\delta K^{-\rho} + (1-\delta)L^{-\rho}\right]\right\}^{-\frac{r}{\rho}} = (j^{-\rho})^{-\frac{r}{\rho}} Q = j^r Q$$

Hence r denotes the degree of homogeneity. With $r > 1$ ($r < 1$), we have increasing (decreasing) returns to scale.

8. Evaluate the following:

(a) $\lim_{x\to 4} \frac{x^2 - x - 12}{x - 4}$ (c) $\lim_{x\to 0} \frac{5^x - e^x}{x}$

(b) $\lim_{x\to 0} \frac{e^x - 1}{x}$ (d) $\lim_{x\to \infty} \frac{\ln x}{x}$

Ans: By L'Hôpital's Rule, we have:

(a) $\lim_{x\to 4} \frac{x^2 - x - 12}{x - 4} = \lim_{x\to 4} \frac{2x - 1}{1} = 7$

(b) $\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \frac{e^x}{1} = 1$

(c) $\lim_{x\to 0} \frac{5^x - e^x}{x} = \lim_{x\to 0} \frac{5^x \ln 5 - e^x}{1} = \ln 5 - 1$

(d) $\lim_{x\to \infty} \frac{\ln x}{x} = \lim_{x\to \infty} \frac{\frac{1}{x}}{1} = 0$

9. By use of L'Hopital's rule, show that

(a) $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$ (b) $\lim_{x\to 0^+} x \ln x = 0$ (c) $\lim_{x\to 0^+} x^x = 1$

Ans:

(a) By successive applications of the rule, we find that

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{nx^{n-1}}{e^x} = \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x} = \ldots = \lim_{x\to\infty} \frac{n!}{e^x} = 0$$

(b) By taking $m(x) = \ln x$, and $n(x) = \frac{1}{x}$, we have

$$\lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0^+} - x = 0$$

(c) Since $x^x = \exp(\ln x^x) = \exp(x \ln x)$, and since, from (b) above, the expression $x \ln x$ tends to zero as $x \to 0^+$, x^x must tend to $e^0 = 1$ as $x \to 0^+$.