CHAPTER 11

EXERCISE 11.2

Use Table 11.1 to find the extreme value(s) of each of the following four functions, and determine whether they are maxima or minima:

1. $z = x^2 + xy + 2y^2 + 3$
2. $z = -x^2 - y^2 + 6x + 2y$
3. $z = ax^2 + by^2 + c$; consider each of the three subcases:
   (a) a>0, b>0 (b) a<0, b<0 (c) a and b opposite in sign
4. $z = e^{2x} - 2x + 2y^2 + 3$

Ans:

1. The derivatives are: $f_x = 2x + y$, $f_y = x + 4y$, $f_{xx} = 2$, $f_{yy} = 4$, and $f_{xy} = 1$. Then first-order condition requires that $2x + y = 0$ and $x + 4y = 0$. Thus we have $x^* = y^* = 0$ implying $z^* = 3$ (which is a minimum)
2. The derivatives are: $f_x = -2x + 6$, $f_y = -2y + 2$, $f_{xx} = -2$, $f_{yy} = -2$, and $f_{xy} = 0$. Then first-order condition requires that $-2x = 6$ and $-2y = -2$. Thus we have $x^* = 3$ $y^* = 1$ so that $z^* = 10$ (which is a maximum)
3. $f_x = 2ax$, $f_y = 2by$, $f_{xx} = 2a$, $f_{yy} = 2b$, and $f_{xy} = 0$. The first-order condition requires that $2ax = 0$ and $2by = 0$. Thus $x^* = y^* = 0$ so that $z^* = c$. The second derivatives give us $f_{xx} f_{yy} = 4ab$, and $f_{xy}^2 = 0$. Thus:
   (a) z^* is a minimum if a,b>0.
   (b) z^* is a maximum if a,b<0.
   (c) z^* gives a saddle point if a and b have opposite signs.
4. $f_x = 2(e^{2x} - 1)$, $f_y = 4y$, $f_{xx} = 4e^{2x}$, $f_{yy} = 4$, and $f_{xy} = 0$. The first-order condition requires that $e^{2x} = 1$ and $4y = 0$. Thus $x^* = y^* = 0$ so that $z^* = 4$. Since $f_{xx} f_{yy} = 4(4)$ exceeds $f_{xy}^2 = 0$, $z^* = 4$ is minimum.

5. Consider the function $z = (x-2)^4 + (y-3)^4$.
   (a) Establish by intuitive reasoning that z attains a minimum $(z^* = 0)$ at $x^* = 2$ and $y^* = 3$.
   (b) Is the first-order necessary condition in Table 11.1 satisfied?
   (c) Is the second-order sufficient condition in Table 11.1 satisfied?
   (d) Find the value of d^2z. Does it satisfy the second-order necessary condition for a minimum in (11.9)?

Ans:

(a) And pair (x, y) other than (2,3) yields a positive z value.

(b) Yes. At $x^* = 2$ and $y^* = 3$, we find $f_x = 4(x-2)^3$ and $f_y = 4(y-3)^3 = 0$.

(c) No. At $x^* = 2$ and $y^* = 3$, we have $f_{xx} = f_{yy} = f_{xy} = f_{yx} = 0$.

(d) By (11.6), $d^2z = 0$. Thus (11.9) is satisfied.

EXERCISE 11.3

1. By direct matrix multiplication, express each of the following matrix products as a quadratic form:

   (a) $\begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$ (c) $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

   (b) $\begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$ (d) $\begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$

Ans:

(a) $q = 4u^2 + 4uv + 3v^2$

(b) $q = -2u^2 + 4uv - 4v^2$

(c) $q = 5x^2 + 6xy$

(d) $q = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2$

2. In Prob.1b and c, the coefficient matrices are not symmetric with respect to the principal diagonal. Verify that by averaging the off-diagonal elements and thus converting them, respectively, into $\begin{bmatrix} -2 & 2 \\ 2 & -4 \end{bmatrix}$ and $\begin{bmatrix} 5 & 3 \\ 3 & 0 \end{bmatrix}$ we will get the same quadratic forms as before.

Ans: For (b): $q = -2u^2 + 4uv - 4v^2$. For (c): $q = 5x^2 + 6xy$. Both are the same as before.

3. On the basis of their coefficient matrices (the symmetric versions), determine by the determinantal test whether the quadratic forms in Prob. 1a, b, and c are either positive definite or negative definite.

Ans:

(a) $\begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}: 4 > 0, 4(3) > 2^2$ -- positive definite

(b) $\begin{bmatrix} -2 & 2 \\ 2 & -4 \end{bmatrix}: -2 < 0, -2(-4) > 2^2$ -- negative definite

(c) $\begin{bmatrix} 5 & 3 \\ 3 & 0 \end{bmatrix}: 5 > 0, 5(0) < 3^2$ -- neither

4. Express each of the following quadratic forms as a matrix product involving a symmetric coefficient matrix:

(a) $q = 3u^2 - 4uv + 7v^2$ (d) $q = 6xy - 5y^2 - 2x^2$

(b) $q = u^2 + 7uv + 3v^2$ (e) $q = 3u_1^2 - 2u_1u_2 + 4u_1u_3 + 5u_2^2 + 4u_3^2 - 2u_2u_3$

(c) $q = 8uv - u^2 - 31v^2$ (f) $q = -u^2 + 4uv - 6uw - 4v^2 - 7w^2$

Ans:

(a) $q = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$

(b) $q = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} 1 & 3.5 \\ 3.5 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$

(c) $q = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 4 & -31 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$

(d) $q = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

(e) $q = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

(f) $q = \begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} -1 & 2 & -3 \\ 2 & -4 & 0 \\ -3 & 0 & -7 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$

5. From the discriminants obtained from the symmetric coefficient matrices of Prob. 4, ascertain by the determinantal test which of the quadratic forms are positive definite and which are negative definite.

Ans:

(a) $3 > 0, 3(7) > (-2)^2$ -- positive definite

(b) $1 > 0, 1(3) < (3.5)^2$ -- neither

(c) $-1 < 0, -1(-31) > 4^2$ -- negative definite

(d) $-2 < 0, -2(-5) > 3^2$ -- negative definite

(e) $3 > 0, \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix} = 14 > 0, \begin{vmatrix} 3 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 4 \end{vmatrix} = 37 > 0$ -- positive definite

(f) $-1 < 0, \begin{vmatrix} -1 & 2 \\ 2 & -4 \end{vmatrix} = 0$ -- neither (no need to check $|D_3|$)

6. Find the characteristic roots of each of the following matrices:

(a) $D = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}$ (b) $E = \begin{bmatrix} -2 & 2 \\ 2 & -4 \end{bmatrix}$ (c) $F = \begin{bmatrix} 5 & 3 \\ 3 & 0 \end{bmatrix}$

What can you conclude about the signs of the quadratic forms $u'Du$, $u'Eu$ and $u'Fu$? (Check your results against Prob.3.)

Ans:

(a) The characteristic equation is

$$\begin{vmatrix} 4-r & 2 \\ 2 & 3-r \end{vmatrix} = r^2 - 7r + 8 = 0$$

Its roots are $r_1, r_2 = \frac{1}{2}(7 + \sqrt{17})$. Both roots being positive, $u'Du$ is positive definite.

(b) The characteristic equation is $r^2 + 6r + 4 = 0$, with roots $r_1, r_2 = -3 \pm \sqrt{5}$. Both roots being negative, $u'Eu$ is negative definite.

(c) The characteristic equation is $r^2 - 5r - 9 = 0$, with roots $r_1, r_2 = \frac{1}{2}(5 \pm \sqrt{61})$. Since r_1 is positive, but r_2 is negative, $u'Fu$ is indefinite.

7. Find the characteristic vectors of the matrix $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$.

Ans: The characteristic equation $\begin{vmatrix} 4-r & 2 \\ 2 & 1-r \end{vmatrix} = r^2 - 5r = 0$ has the roots $r_1 = 5$ and $r_2 = 0$. (Note: This is an example where $|D| = 0$). Using r_1 in (11.13'), we have $\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$. Thus $x_1 = 2x_2$. Upon normalization, we obtain the first characteristic vector $v_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$. Next, using r_2 in (11.13'), we have $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$. Therefore, $x_1 = -\frac{1}{2}x_2$. Upon normalization, we obtain $v_2 = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$. These results happen to be identical with those in Example 5.

8. Given a quadratic form $u'Du$, where D is 2×2, the characteristic equation of D can be written as

$$\begin{vmatrix} d_{11} - r & d_{12} \\ d_{21} & d_{22} - r \end{vmatrix} = 0 \qquad (d_{12} = d_{21})$$

Expand the determinant; express the roots of this equation by use of the quadratic formula; and deduce the following:

(a) No imaginary number (a number involving $\sqrt{-1}$) can occur in r_1 and r_2.

(b) To have repeated roots, matrix D must be in the form of $\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$.

(c) To have either positive or negative semidefiniteness, the discriminant of the quadratic form may vanish, that is, $|D| = 0$ is possible.

Ans: The characteristic equation can be written as

$$r^2 - (d_{11} + d_{22})r + (d_{11}d_{22} - d_{12}d_{21}) = 0$$

Thus $r_1, r_2 = \frac{1}{2}[(d_{11} + d_{22}) \pm \sqrt{(d_{11} + d_{22})^2 - 4(d_{11}d_{22} - d_{12}d_{21})}]$

(a) The expression under the square-root sign can be written as

$$E = d_{11}^2 + 2d_{11}d_{22} + d_{22}^2 - 4d_{11}d_{22} + 4d_{12}d_{21}$$
$$= d_{11}^2 - 2d_{11}d_{22} + d_{22}^2 + 4d_{12}d_{21} = (d_{11} - d_{22})^2 + 4d_{12}^2 \geq 0$$

Thus no imaginary number can occur in r_1 and r_2.

(b) To have repeated roots, E has to be zero, which can occur if and only if $d_{11} = d_{22}$ (say, =c) and at the same time $d_{12} = d_{21} = 0$. This would mean that matrix D takes the form of $\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$.

(c) Positive or negative semidefiniteness allows a characteristic root to be zero (r=0), which implies the possibility that the characteristic equation reduces to $d_{11}d_{22} - d_{12}d_{21} = 0$, or $|D| = 0$.

EXERCISE 11.4

Find the extreme values, if any, of the following four functions. Check whether they are maxima or minima by the determinantal test.

1. $z = x_1^2 + 3x_2^2 - 3x_1x_2 + 4x_2x_3 + 6x_3^2$

2. $z = 29 - (x_1^2 + x_2^2 + x_3^2)$

3. $z = x_1x_3 + x_1^2 - x_2 + x_2x_3 + x_2^2 + 3x_3^2$

4. $z = e^{2x} + e^{-y} + e^{w^2} - (2x + 2e^w - y)$

Then answer the following questions regarding Hessian matrices and their characteristic roots.

Ans:

1. The first-order condition

$$f_1 = 2x_1 - 3x_2 = 0$$
$$f_2 = -3x_1 + 6x_2 + 4x_3 = 0$$
$$f_3 = 4x_2 + 12x_3 = 0$$

is a homogeneous linear-equation system in which the three equations are independent. Thus the only solution is

$x_1^* = x_2^* = x_3^* = 0$ so that $z^* = 0$

The Hessian is $\begin{vmatrix} 2 & -3 & 0 \\ -3 & 6 & 4 \\ 0 & 4 & 12 \end{vmatrix}$, with $|H_1| = 2 > 0$, $|H_2| = 3 > 0$, and

$|H_3| = 4 > 0$. Consequently, $z^* = 0$ is a minimum.

2. The first-order condition consists of the three equations
$f_1 = -2x_1 = 0 \quad f_2 = -2x_2 = 0 \quad f_3 = -2x_3 = 0$

Thus $x_1^* = x_2^* = x_3^* = 0$ so that $z^* = 29$

The Hessian is $\begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix}$, with $|H_1| = -2 < 0$, $|H_2| = 4 > 0$, and

$|H_3| = -8 < 0$. Consequently, $z^* = 29$ is a maximum.

3. The three equations in the first-order conditions are
$2x_1 + x_3 = 0$
$2x_2 + x_3 = 1$
$x_1 + x_2 + 6x_3 = 0$

Thus $x_1^* = \frac{1}{20}$ $x_2^* = \frac{11}{20}$ $x_3^* = -\frac{2}{20}$ so that $z^* = -\frac{11}{40}$. Since the Hessian is

$\begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 6 \end{vmatrix}$, with $|H_1| = 2 > 0$, $|H_2| = 4 > 0$, and $|H_3| = 20 > 0$, the z^* value

is a minimum.

4. By the first-order condition, we have

$f_x = 2e^{2x} - 2 = 0$, $f_y = -e^{-y} + 1 = 0$, $f_w = 2we^{w^2} - 2e^w = 0$

Thus $x^* = 0$ $y^* = 0$ $w^* = 1$ so that $z^* = 2 - e$
Note: The value of x^* and y^* are found from the fact that $e^0 = 1$. Finding w^* is more complicated. One way of doing it is as follows: First, rewrite the equation $f_w = 0$ as $we^{w^2} e^w$. Taking natural logs yield

or $\ln w + \ln e^{w^2} = \ln e^w$

or $\ln w + w^2 = w$

or $\ln w = w - w^2$

If we draw a curve for $\ln w$, and another for $w - w^2$, their intersection point will give us the solution. The $\ln w$ curve is a strictly concave curve with horizontal intercept at $w = 1$. The $w - w^2$ is a hill-type parabola with horizontal intercepts $w = 0$ and $w = 1$. Thus the solution is $w^* = 1$.

The Hessian is $\begin{vmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4e \end{vmatrix}$ when evaluated at the stationary point, with all leading principal minors positive. Thus z^* is a minimum.

5. (a) Which of Prob. 1 through 4 yield diagonal Hessian matrices? In each such case, do the diagonal elements possess a uniform sign?
   (b) What can you conclude about the characteristic roots of each diagonal Hessian matrix found? About the sign definiteness of d^2z?
   (c) Do the results of the characteristic-root test check with those of the determinantal test?

Ans:

(a) Problems 2 and 4 yield diagonal Hessian matrices. The diagonal elements are all negative fro problem 2, and all positive for problems 4 and 5.

(b) According to (11.16), these diagonal elements represent the characteristic roots. Thus the characteristic roots are all negative (d^2z negative definite) for problem 2, and all positive (d^2z positive definite) for problem 4.

(c) Yes.

6. (a) Find the characteristic roots of the Hessian matrix for Prob. 3.
   (b) What can you conclude from your results?
   (c) Is your answer to (b) consistent with the result of t he determinantal test for Prob. 3?

Ans:

(a) The characteristic equation is, by (11.14):

$$\begin{vmatrix} 2-r & 0 & 1 \\ 0 & 2-r & 1 \\ 1 & 1 & 6-r \end{vmatrix} = 0$$

Expanding the determinant by the method of Fig. 5.1, we get

$$(2-r)(2-r)(6-r)-(2-r)-(2-r)=0$$

or $\quad (2-r)[(2-r)(6-r)-2]=0 \quad$ [factoring]

or $\quad (2-r)(r^2-8r+10)=0$

Thus, from the $(2-r)$ term, we have $r_1=2$. By the quadratic formula, we get from the other term: $r_2, r_3 = 4 \pm \sqrt{6}$.

(b) All three roots are positive. Thus d^2z is positive definite, and z^* is a minimum.

(c) Yes.

EXERCISE 11.5

1. Use (11.20) to check whether the following functions are concave, convex, strictly concave, strictly convex, or neither:

   (a) $z=x^2$ (b) $z=x_1^2+2x_2^2$ (c) $z=2x^2-xy+y^2$

Ans:

(a) Let u and v be any two distinct points in the domain. Then

$$f(u)=u^2 \qquad f(v)=v^2 \qquad f[\theta u+(1-\theta)v]=[\theta u+(1-\theta)v]^2$$

Substituting these into (11.20), we find the difference between the left- and right-side expressions in (11.20) to be

$$\theta u^2+(1-\theta)v^2-\theta^2u^2-2\theta(1-\theta)uv-(1-\theta)^2v^2$$
$$=(1-\theta)u^2-2\theta(1-\theta)uv+\theta(1-\theta)v^2$$
$$=(1-\theta)(u-v)^2>0 \quad [\text{since } u\neq v]$$

Thus $z=x^2$ is strictly convex function.

(b) Let $u=(u_1,u_2)$ and $v=(v_1,v_2)$ be any two distinct points in the domain. Then

$$f(u)=u_1^2+2u_2^2 \qquad f(v)=v_1^2+2v_2^2$$
$$f[\theta u+(1-\theta)v]=[\theta u_1+(1-\theta)v_1]^2+2[\theta u_2+(1-\theta)v_2]^2$$

The difference between the left- and right-side expressions in (11.20) is

$$\theta(1-\theta)(u_1^2-2u_1v_1+v_1^2+2u_2^2-4u_2v_2+2v_2^2)=\theta(1-\theta)[(u_1-v_1)^2+2(u_2-v_2)^2]>0$$

Thus $z=x_1^2+2x_2^2$ is a strictly convex function.

(c) Let $u=(u_1,u_2)$ and $v=(v_1,v_2)$ be any two distinct points in the domain. Then

$$f(u)=2u_1^2-u_1u_2+u_2 \qquad f(v)=2v_1^2-v_1v_2+v_2^2$$
$$f[\theta u+(1-\theta)v]=2[\theta u_1+(1-\theta)v_1^2]-[\theta u_1+(1-\theta)v_1]\cdot[\theta u_2+(1-\theta)v_2]+[\theta u_2+(1-\theta)v_2]^2$$

The difference between the left- and right-side expressions in (11.20) is

$$\theta(1-\theta)[(2u_1^2-4u_1v_1+2v_1^2)-u_1u_2+u_1v_2+v_1u_2-v_1v_2+(u_2^2-2u_2v_2+v_2^2)]$$

$= \theta(1-\theta)[2(u_1 - v_1)^2 - (u_1 - v_1)(u_2 - v_2) + (u_2 - v_2)^2] > 0$

because the bracketed expression is positive, like $\theta(1-\theta)$. [The bracketed expression, a positive-definite quadratic form in the two variables $(u_1 - v_1)$ and $(u_2 - v_2)$, is positive since $(u_1 - v_1)$ and $(u_2 - v_2)$ are not both zero in our problem.] Thus $z = 2x^2 - xy + y^2$ is strictly convex function.

2. Use (11.24) or (11.24') to check whether the following functions are concave, convex, strictly concave, strictly convex, or neither:

   (a) $z = -x^2$ (b) $z = (x_1 + x_2)^2$ (c) $z = -xy$

Ans:

(a) With $f'(u) = -2u$, the difference between the left- and right-side expressions in (11.24) is

$-v^2 + u^2 + 2u(v-u) = -v^2 + 2uv - u^2 = -(v-u)^2 < 0$

Thus $z = -x^2$ is strictly concave.

(b) Since $f_1(u_1, u_2) = f_2(u_1, u_2) = 2(u_1 + u_2)$, the difference between the left- and right-side expressions in (11.24') is

$(v_1 + v_2)^2 - (u_1 + u_2)^2 - 2(u_1 + u_2)[(v_1 - u_1) + (v_2 - u_2)]$

$= (v_1 + v_2)^2 - 2(v_1 + v_2)(u_1 + u_2) + (u_1 + u_2)^2$

$= [(v_1 + v_2) - (u_1 + u_2)]^2 \geq 0$

A zero value cannot be ruled out because the two points may be, e.g., $(u_1, u_2) = (5,3)$ and $(v_1, v_2) = (2,6)$. Thus $z = (x_1 + x_2)^2$ is convex, but not strictly so.

(c) Since $f_1(u_1, u_2) = -u_2$, and $f_2(u_1, u_2) = -u_1$, the difference between the left- and right-side expressions in (11.24') is

$-v_1 v_2 + u_1 u_2 + u_2(v_1 - u_1) + u_1(v_2 - u_2)$

$= -v_1 v_2 + v_1 u_2 + u_1 v_2 - u_1 u_2 = (v_1 - u_1)(v_2 - u_2) \gtrless 0$

Thus $z = -xy$ is neither convex nor concave.

3. In view of your answer to Prob. 2c, could you have made use of Theorem III of this section to compartmentalize the task of checking the function $z = 2x^2 - xy + y^2$ in Prob. 1c? Explain your answer.

Ans: No. That theorem gives a sufficient condition which is not satisfied.

4. Do the following constitute convex sets in the 3-space?

   (a) A doughnut (b) A bowling pin (c) A perfect marble

Ans: (a) No. (b) No. (c) Yes.

5. The equation $x^2 + y^2 = 4$ represents a circle with center at (0,0) and with a radius of 2.

   (a) Interpret geometrically the set $\{(x, y) | x^2 + y^2 \le 4\}$.

   (b) Is this set convex?

Ans:

(a) The circle with its interior, i.e. a disk.

(b) Yes.

6. Graph each of the following sets, and indicate whether it is convex:

   (a) $\{(x, y) | y = e^x\}$

   (b) $\{(x, y) | y \ge e^x\}$

   (c) $\{(x, y) | y \le 13 - x^2\}$

   (d) $\{(x, y) | xy \ge 1; x > 0, y > 0\}$

Ans:

(a) The set of points on an exponential curve; not a convex set.

(b) The set of points lying on or above an exponential curve; a convex set.

(c) The set of points lying on or below an inverse U-shaped curve; a convex set.

(d) The set of points lying on or above a rectangular hyperbola in the positive quadrant; a convex set.

7. Given $u = \begin{bmatrix} 10 \\ 6 \end{bmatrix}$ and $v = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$, which of the following are convex combinations of u and v?

   (a) $\begin{bmatrix} 7 \\ 7 \end{bmatrix}$ (b) $\begin{bmatrix} 5.2 \\ 7.6 \end{bmatrix}$ (c) $\begin{bmatrix} 6.2 \\ 8.2 \end{bmatrix}$

Ans:

(a) This is a convex combination, with $\theta = 0.5$.

(b) This is again a convex combination, with $\theta = 0.2$.

(c) This is not a convex combination.

8. Given two vectors u and v in the 2-space, find and sketch:
   (a) The set of all linear combinations of u and v.
   (b) The set of all nonnegative linear combinations of u and v.
   (c) The set of all convex combinations of u and v.

Ans:

(a) This set is the entire 2-space.

(b) This set is a cone bounded on one side by a ray passing through point u, and on the other side by a ray passing through point v.

(c) This set is the line segment uv.

9. (a) Rewrite (11.27) and (11.28) specifically for the cases where the f and g functions have n independent variables.
   (b) Let $n = 2$, and let the function f be shaped like a (vertically held) ice-cream cone whereas the function g is shaped like a pyramid. Describe the sets $S^{\leq}$ and $S^{\geq}$.

Ans:

(a) $S^{\leq} \equiv \left\{(x_1, \ldots, x_n \middle| f(x_1, \ldots, x_n) \leq k\right\}$ (f convex)

$S^{\geq} \equiv \left\{(x_1, \ldots, x_n \middle| g(x_1, \ldots, x_n) \geq k\right\}$ (g concave)

(b) $S^{\leq}$ is a solid circle (or disk); $S^{\geq}$ is a solid square.

EXERCISE 11.6

1. If the competitive firm of Example 1 has the cost function $C = 2Q_1^2 + 2Q_2^2$ instead, then:
   (a) Will the production of the two goods still be technically related?
   (b) What will be the new optimal levels of Q_1 and Q_2?
   (c) What is the value of π_{12}? What does this imply economically?

Ans:

(a) No, because the marginal cost of one commodity will be independent of the output of the other.

(b) The first-order condition is

$\pi_1 = P_{10} - 4Q_1 = 0 \qquad \pi_2 = P_{20} - 4Q_2 = 0$

Thus $Q_1^* = \frac{1}{4}P_{10}$ and $Q_2^* = \frac{1}{4}P_{20}$. The profit is maximized, because the

Hessian is $\begin{vmatrix} -4 & 0 \\ 0 & -4 \end{vmatrix}$, with $|H_1| < 0$ and $|H_2| > 0$. The signs of the principal minors do not depend on where they are evaluated. Thus the maximum in this problem is a unique absolute maximum.

(c) $\pi_{12} = 0$ implies that the profit-maximizing output level of one commodity is independent of the output of the other (see first-order condition). The firm can operate as if it has two plants, each optimizing the output of a different product.

2. A two-product firm faces the following demand and cost functions:

$Q_1 = 40 - 2P_1 - P_2 \qquad Q_2 = 35 - P_1 - P_2 \qquad C = Q_1^2 + 2Q_2^2 + 10$

(a) Find the output levels that satisfy the first-order condition for maximum profit. (Use fractions.)

(b) Check the second-order sufficient condition. Can you conclude that this problem possesses a unique absolute maximum?

(c) What is the maximal profit?

Ans:

(a) By the procedure used in Example 2 (taking Q_1 and Q_2 as choice variables), we can find $Q_1^* = 3\frac{4}{7}$, $Q_2^* = 4\frac{9}{14}$, $P_1^* = 6\frac{1}{14}$, $P_2^* = 24\frac{2}{7}$.

(b) The Hessian is $\begin{vmatrix} -4 & 2 \\ 2 & -8 \end{vmatrix}$, with $|H_1| = -4$ and $|H_2| = 28$. Thus the sufficient condition for a maximum is met.

(c) Substituting the P^*'s and Q^*'s into the R and C functions, we get

$R^* = 134\frac{43}{98}$, $C^* = 65\frac{85}{98}$, and $\bar{r} = 68\frac{4}{7}$

3. On the basis of the equilibrium price and quantity in Example 4, calculate the point elasticity of demand $|\varepsilon_{di}|$ (for $i = 1,2$). Which market has the highest and the lowest demand elasticities?

Ans: $|c_{d1}| = \left|\frac{dQ_1}{dP_1}\frac{P_1^*}{Q_1^*}\right| = \frac{1}{4}\frac{39}{6} = \frac{13}{8}$. Similarly, $|c_{d2}| = \frac{1}{5}\frac{60}{9} = \frac{4}{3}$, and $|c_{d3}| = \frac{1}{6}\frac{45}{5} = \frac{3}{2}$.

The highest is $|c_{d1}|$; the lowest is $|c_{d2}|$.

4. If the cost function of Example 4 is changed to $C = 20 + 15Q + Q^2$
   (a) Find the new marginal-cost function.
   (b) Find the new equilibrium quantities. (Use fractions.)
   (c) Find the new equilibrium prices.
   (d) Verify that the second-order sufficient condition is met.

Ans:

(a) $C' = 15 + 2Q = 15 + 2Q_1 + 2Q_2 + 2Q_3$

(b) Equating each MR to the MC, we obtain the three equations:
$10Q_1 + 2Q_2 + 2Q_3 = 48$ $\quad 2Q_1 + 12Q_2 + 2Q_3 = 90$ and
$2Q_1 + 2Q_2 + 14Q_3 = 60$

Thus $Q_1^* = 2\frac{88}{97}$, $Q_2^* = 6\frac{51}{97}$, $Q_3^* = 2\frac{91}{97}$.

(c) Substituting the above into the demand equations, we get

$P_1^* = 51\frac{36}{97}$, $P_2^* = 72\frac{36}{97}$, $P_3^* = 57\frac{36}{97}$

(d) Since $R_1'' = -8$, $R_2'' = -10$, $R_3'' = -12$, and $C'' = 2$, we do find that: (1) $R_1'' - C'' = -10$ (2) $R_1''R_2'' - (R_1'' + R_2'')C'' = 80 + 36 = 116 > 0$, and (3)

$|H| = -960 - (80 + 96 + 120)(2) = -1552 < 0$

5. In Example 7, how would you rewrite the profit function if the following conditions hold?
   (a) Interest is compounded semiannually at an interest rate of i_0 per annum, and the production process takes 1 year.
   (b) Interest is compounded quarterly at an interest rate of i_0 per annum, and the production process takes 9 months.

Ans:

(a) $\pi = P_0 Q(a,b)(1 + \frac{1}{2} i_0)^{-2} - P_{a0}a - P_{b0}b$

(b) $\pi = P_0 Q(a,b)(1 + \frac{1}{4} i_0)^{-3} - P_{a0}a - P_{b0}b$

6. Given $Q = Q(a,b)$, how would you express algebraically the isoquant for the output level of, say, 260?

Ans: $Q(a,b) = 260$

EXERCISE 11.7

For Probs. 1 through 3, assume that $Q_{ab} > 0$.

1. On the basis of the model described in (11.45) through (11.48), find the comparative-static derivatives $(\partial a^*/\partial P_{a0})$ and $(\partial b^*/\partial P_{a0})$. Interpret the economic meaning of the result. Then analyze the effects on a^* and b^* of a change in P_{b0}.

Ans:

(a) We may take (11.49) as the point of departure. Letting P_{a0} alone vary (i.e., letting $dP_0 = dP_{b0} = dr = dt = 0$), and dividing through by $dP_{a0} \neq 0$, we get the matrix equation

$$\begin{bmatrix} P_0 Q_{aa} e^{-rt} & P_0 Q_{ab} e^{-rt} \\ P_0 Q_{ab} e^{-rt} & P_0 Q_{bb} e^{-rt} \end{bmatrix} \begin{bmatrix} \left(\frac{\partial a^*}{\partial P_{a0}}\right) \\ \left(\frac{\partial b^*}{\partial P_{a0}}\right) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Hence, by Cramer's Rule,

$$\left(\frac{\partial a^*}{\partial P_{a0}}\right) = \frac{P_0 Q_{bb} e^{-rt}}{|J|} < 0 \quad \text{and} \quad \left(\frac{\partial b^*}{\partial P_{a0}}\right) = -\frac{P_0 Q_{bb} e^{-rt}}{|J|} < 0$$

The higher the price of input a, the smaller will be the equilibrium levels of inputs a and b.

(b) Next, letting P_{b0} alone vary in (11.49), and dividing through by $dP_{b0} \neq 0$, we can obtain results similar to (a) above:

$$\left(\frac{\partial a^*}{\partial P_{b0}}\right) = -\frac{P_0 Q_{ab} e^{-rt}}{|J|} < 0 \quad \text{and} \quad \left(\frac{\partial b^*}{\partial P_{b0}}\right) = \frac{P_0 Q_{aa} e^{-rt}}{|J|} < 0$$

2. For the problem of Example 7 in Sec. 11.6:
   (a) How many parameters are there? Enumerate them.
   (b) Following the procedure described in (11.45) through (11.50), and assuming that the second-order sufficient condition is satisfied, find the comparative-static derivatives $(\partial a^*/\partial P_0)$ and $(\partial b^*/\partial P_0)$. Evaluate their signs and interpret their economic meanings.
   (c) Find $(\partial a^*/\partial i_0)$ and $(\partial b^*/\partial i_0)$, evaluate their signs, and interpret their economic meanings.

Ans:

(a) P_0, i_0, P_{a0}, P_{b0}.

(b) From the first-order condition, we can check the Jacobian

$$|J| = \begin{vmatrix} \frac{\partial F^1}{\partial a} & \frac{\partial F^1}{\partial b} \\ \frac{\partial F^2}{\partial a} & \frac{\partial F^2}{\partial b} \end{vmatrix} = \begin{vmatrix} P_0 Q_{aa}(1+i_0)^{-1} & P_0 Q_{ab}(1+i_0)^{-1} \\ P_0 Q_{ab}(1+i_0)^{-1} & P_0 Q_{bb}(1+i_0)^{-1} \end{vmatrix} = P_0^2 (1+i_0)^{-2} \begin{vmatrix} Q_{aa} & Q_{ab} \\ Q_{ab} & Q_{bb} \end{vmatrix}$$

the be positive at the initial equilibrium (optimum) since the second-order sufficient condition is assumed to be satisfied. By the implicit-function theorem, we can then write

$a^* = a^*(P_0, i_0, P_{a0}, P_{b0})$ and $b^* = b^*(P_0, i_0, P_{a0}, P_{b0})$

we can also write the identities

$$P_0 Q_a(a^*, b^*)(1+i_0)^{-1} - P_{a0} \equiv 0$$

$$P_0 Q_a(a^*, b^*)(1+i_0)^{-1} - P_{b0} \equiv 0$$

Taking the total differentials, we get (after rearrangement) the following pair of equations corresponding to (11.49):

$$P_0 Q_{aa}(1+i_0)^{-1} da^* + P_0 Q_{ab}(1+i_0)^{-1} db^* = -Q_a(1+i_0)^{-1} dP_0 + P_0 Q_a (1+i_0)^{-2} di_0 + dP_{a0}$$

$$P_0 Q_{ab}(1+i_0)^{-1} da^* + P_0 Q_{bb}(1+i_0)^{-1} db^* = -Q_b(1+i_0)^{-1} dP_0 + P_0 Q_b (1+i_0)^{-2} di_0 + dP_{b0}$$

Letting P_0 alone vary (i.e., letting $di_0 = dP_{a0} dP_{b0} = 0$), and dividing through by $dP_0 \neq 0$, we get

$$\begin{bmatrix} P_0 Q_{aa}(1+i_0)^{-1} & P_0 Q_{ab}(1+i_0)^{-1} \\ P_0 Q_{ab}(1+i_0)^{-1} & P_0 Q_{bb}(1+i_0)^{-1} \end{bmatrix} \begin{bmatrix} \frac{\partial a^*}{\partial P_0} \\ \frac{\partial b^*}{\partial P_0} \end{bmatrix} = \begin{bmatrix} -Q_a(1+i_0)^{-1} \\ -Q_b(1+i_0)^{-1} \end{bmatrix}$$

Thus $\left(\frac{\partial a^*}{\partial P_0}\right) = \frac{(Q_b Q_{ab} - Q_a Q_{bb}) P_0 (1+i_0)^{-2}}{|J|} > 0$

$$\left(\frac{\partial b^*}{\partial P_0}\right) = \frac{(Q_a Q_{ab} - Q_b Q_{aa}) P_0 (1+i_0)^{-2}}{|J|} > 0$$

(c) Letting i_0 alone vary, we can similarly obtain

$$\begin{bmatrix} P_0 Q_{aa}(1+i_0)^{-1} & P_0 Q_{ab}(1+i_0)^{-1} \\ P_0 Q_{ab}(1+i_0)^{-1} & P_0 Q_{bb}(1+i_0)^{-1} \end{bmatrix} \begin{bmatrix} \frac{\partial a^*}{\partial i_0} \\ \frac{\partial b^*}{\partial i_0} \end{bmatrix} = \begin{bmatrix} P_0 Q_a(1+i_0)^{-2} \\ P_0 Q_b(1+i_0)^{-2} \end{bmatrix}$$

Thus $\left(\frac{\partial a^*}{\partial i_0}\right) = \frac{(Q_a Q_{bb} - Q_b Q_{ab}) P_0^2 (1+i_0)^{-3}}{|J|} > 0$

$$\left(\frac{\partial b^*}{\partial i_0}\right) = \frac{(Q_b Q_{aa} - Q_a Q_{ab}) P_0 (1+i_0)^{-3}}{|J|} > 0$$

3. Show that the results in (11.50) can be obtained alternatively by differentiating the two identities in (11.48) totally with respect to P_0, while holding the other exogenous variables fixed. Bear in mind that P_0 can affect a^* and b^* by virtue of (11.47).

Ans: Differentiating (11.49) totally respect to P_0, we get

$$Q_a e^{-rt} + P_0 Q_{aa}\left(\tfrac{\partial a^*}{\partial P_0}\right)e^{-rt} + P_0 Q_{ab}\left(\tfrac{db^*}{dP_0}\right)e^{-rt} = 0$$

$$Q_b e^{-rt} + P_0 Q_{ab}\left(\tfrac{\partial a^*}{\partial P_0}\right)e^{-rt} + P_0 Q_{bb}\left(\tfrac{db^*}{dP_0}\right)e^{-rt} = 0$$

Or, in matrix notation,

$$\begin{bmatrix} P_0 Q_{aa} e^{-rt} & P_0 Q_{ab} e^{-rt} \\ P_0 Q_{ab} e^{-rt} & P_0 Q_{bb} e^{-rt} \end{bmatrix} \begin{bmatrix} \left(\partial a^* / \partial P_0\right) \\ \left(\partial b^* / \partial P_0\right) \end{bmatrix} = \begin{bmatrix} -Q_a e^{-rt} \\ -Q_b e^{-rt} \end{bmatrix}$$

which leads directly to the results in (11.50).

4. A Jacobian determinant, as defined in (7.27), is made up of first-order partial derivatives. On the other hand, a Hessian determinant, as defined in Sec. 11.3 and 11.4, has as its elements second-order partial derivatives. How, then, can it turn out that $|J| = |H|$, as in (11.46)?

Ans: In (11.46), the elements of the Jacobian determinant are first-order partial derivatives of the components of the first-order condition shown in (11.45). Thus, those elements are really the second-order partial derivatives of the (primitive) objective function – exactly what are used to construct the Hessian determinant.