CHAPTER 10

EXERCISE 10.1

1. Plot in a single diagram the graphs of the exponential function $y = 3^t$ and $y = 3^{2t}$.
   (a) Do the two graphs display the same general positional relationship as shown in Fig. 10.2a?
   (b) Do these two curves share the same y intercept? Why?
   (c) Sketch the graph of the function $y = 3^{3t}$ in the same diagram.

Ans:
(a) Yes.
(b) Yes, because at $t = 0$, the value of y for the two functions are identical: $3^0 = 1$, and $3^{2(0)} = 1$.
(c) N/A.

2. Plot in a single diagram the graphs of the exponential functions $y = 4^t$ and $y = 3(4^t)$.
   (a) Do the two graphs display the general positional relationship suggested in Fig. 10.2b?
   (b) Do these two curves have the same y intercept? Why?
   (c) Sketch the graph of the function $y = \frac{3}{2}(4^t)$ in the same diagram.

Ans:
(a) Yes.
(b) No, because at $t = 0$, the value of y for the two functions are unequal: $4^0 = 1$, but $3(4^0) = 3$.
(c) N/A.

3. Taking for granted that e^t is its own derivative, use the chain rule to find dy/dt for the following:
   (a) $y = e^{5t}$ (b) $y = 4e^{3t}$ (c) $y = 6e^{-2t}$

Ans:
(a) Let $w = 5t$ (so that $dw/dt = 5$), then $y = e^w$ and $dy/dw = e^w$. Thus, by

the chain rule, $\frac{dy}{dt} = \frac{dy}{dw}\frac{dw}{dt} = 5e^{w} = 5e^{5t}$.

(b) Let $w = 3t$, then $y = 4e^{w}$ and $dy/dw = 4e^{w}$. Thus, we have $\frac{dy}{dt} = \frac{dy}{dw}\frac{dw}{dt} = 13e^{3t}$.

(c) Similarly to (b) above, $dy/dt = -12e^{-2t}$.

4. In view of our discussion about (10.1), do you expect the function $y = e^{t}$ to be strictly increasing at an increasing rate? Verify your answer by determining the signs of the first and second derivatives of this function. In doing so, remember that the domain of this function is the set of all real numbers, i.e., the interval $(-\infty, \infty)$.

Ans: The first two derivatives are $y'(t) = y''(t) = e^{t} = (2.718)^{t}$. The value of t can be either positive, zero, or negative. If t>0, then e^{t} is clearly positive; if $t = 0$, then $e^{t} = 1$, again positive; finally, if t<0, say $t = -2$, then $e^{t} = 1/(2.718)^{2}$, still positive. Thus $y'(t)$ and $y''(t)$ are always positive, and the function $y = e^{t}$ always increases at an increasing rate.

5. In (10.2), if negative values are assigned to a and c, the general shape of the curves in Fig. 10.2 will no longer prevail. Examine the change in curve configuration by contrasting (a) the case of $a = -1$ against the case of $a = 1$, and (b) the case of $c = -1$ against the case of $c = 1$.

Ans:

(a) The curve with $a = -1$ is the mirror image of the curve with $a = 1$ with reference to the horizontal axis.

(b) The curve with $c = -1$ is the mirror image of the curve with $c = 1$ with reference to the vertical axis.

EXERCISE 10.2

1. Use the infinite-series form of e^{x} in (10.6) to find the approximate value of:

   (a) e^{2} (b) $\sqrt{e}$ $(= e^{1/2})$

   (Round off your calculation of each term to three decimal places, and continue with the series till you get a term 0.000.)

Ans:

(a) $e^2 = 1 + 2 + \frac{1}{2}(2)^2 + \frac{1}{6}(2)^3 + \frac{1}{24}(2)^4 + \frac{1}{120}(2)^5 + \frac{1}{720}(2)^6 + \frac{1}{5040}(2)^7 + \frac{1}{40320}(2)^8$

$$= + \frac{1}{362880}(2)^9 + \frac{1}{3628800}(2)^{10}$$

$$= 1 + 2 + 2 + 1.333 + 0.667 + 0.267 + 0.089 + 0.025 + 0.006 + 0.001 + 0.000$$
$$= 7.388$$

(b) $e^{1/2} = 1 + \frac{1}{2} + \frac{1}{2}(\frac{1}{2})^2 + \frac{1}{6}(\frac{1}{2})^3 + \frac{1}{24}(\frac{1}{2})^4 + \frac{1}{120}(\frac{1}{2})^5 = 1 + 0.5 + 0.125 + 0.021 + 0.003 + 0.000$

$$= 1.649$$

2. Given the function $\phi(x) = e^{2x}$:

   (a) Write the polynomial part P_n of its Maclaurin Series.

   (b) Write the Lagrange form of the remainder R_n. Determine whether $R_n \to 0$ as $n \to \infty$, that is, whether the series is convergent to $\phi(x)$.

   (c) If convergent, so that $\phi(x)$ may be expressed as an infinite series, write out this series.

Ans:

(a) The derivatives are: $\phi' = 2e^{2x}$, $\phi'' = 2^2 e^{2x}$, $\phi''' = 2^3 e^{2x}$, or in general $\phi^{(k)} = 2^k e^{2x}$. Thus we have $\phi'(0) = 2$, $\phi''(0) = 2^2$, or more generally $\phi^{(k)}(0) = 2^k$. Accordingly,

$$P_n = 1 + 2x + \frac{1}{2!}2^2 x^2 + \frac{1}{3!}2^3 x^3 + \cdots + \frac{1}{n!}2^n x^n = 1 + 2x + \frac{1}{2!}(2x)^2 + \frac{1}{3!}(2x)^3 + \cdots + \frac{1}{n!}(2x)^n$$

(b) $R_n = \frac{\phi^{(n+1)}(p)}{(n+1)!}x^{n+1} = \frac{2^{(n+1)}e^{2p}}{(n+1)!}x^{n+1} = \frac{e^{2p}}{(n+1)!}(2x)^{n+1}$

It can be verified that $R_n \to 0$ as $n \to \infty$.

(c) Hence $\phi(x)$ can be expressed as an infinite series:

$$\phi(x) = 1 + 2x + \frac{1}{2!}(2x)^2 + \frac{1}{3!}(2x)^3 + \cdots$$

3. Write an exponential expression for the value:

   (a) \$70, compounded continuously at the interest rate of 4% for 3 years

(b) \$690, compounded continuously at t he interest rate of 5% for 2 years
(These interest rates are nominal rates per annum.)

Ans: (a) $\$70e^{0.04(3)} = \$70e^{0.12}$ (b) $\$690e^{0.05(2)} = \$690e^{0.10}$

4. What is the instantaneous rate of growth of y in each of the following?
   (a) $y = e^{0.07t}$ (c) $y = Ae^{0.4t}$
   (b) $y = 15e^{0.03t}$ (d) $y = 0.03e^{t}$

Ans: (a) 0.07 (or 7%) (b) 0.03 (c) 0.40 (d) 1 (or 100%)

5. Show that the two function $y_1 = Ae^{rt}$ (interest compounding) and $y_2 = Ae^{-rt}$ (discounting) are mirror images of each other with reference to the y axis [cf. Exercise 10.1-5, part (b)].

Ans: When $t = 0$, the two functions have that same value (the same y intercept). Also, $y_1 = Ae^{r}$ when $t = 1$, but $y_2 = Ae^{r}$ when $t = -1$. Generally, $y_1 = y_2$ whenever the value of t in one function is the negative of the t value in the other; hence the mirror-image relationship.

EXERCISE 10.3

1. What are the values of the following logarithms?
   (a) $\log_{10} 10{,}000$ (c) $\log_3 81$
   (b) $\log_{10} 0.0001$ (d) $\log_5 3{,}125$

Ans: (a) 4 (b) –4 (c) 4 (d) 5

2. Evaluate the following:
   (a) $\ln e^7$ (c) $\ln(1/e^3)$ (e) $(e^{\ln 3})!$
   (b) $\log_e e^{-4}$ (d) $\log_e(1/e^2)$ (f) $\ln e^x - e^{\ln x}$

Ans: (a) 7 (b) –4 (c) –3 (d) –2 (e) 6 (f) 0

3. Evaluate the following by application of the rules of logarithms:
   (a) $\log_{10}(100)^{13}$ (c) $\ln(3/B)$ (e) $\ln ABe^{-4}$
   (b) $\log_{10} \frac{1}{100}$ (d) $\ln Ae^2$ (f) $(\log_4 e)(\log_e 64)$

Ans:

(a) $\log_{10}(100)^{13} = 13\log_{10} 100 = 13(2) = 26$

(b) $\log_{10}(\frac{1}{100}) = \log_{10} 1 - \log_{10} 100 = 0 - 2 = -2$

(c) $\ln \frac{3}{B} = \ln 3 - \ln B$

(d) $\ln Ae^2 = \ln A = \ln e^2 = \ln A + 2$

(e) $\ln ABe^{-4} = \ln A + \ln B + \ln e^{-4} = \ln A + \ln B - 4$

(f) $(\log_4 e)(\log_e 64) = \log_4 64 = 3$

4. Which of the following are valid?

(a) $\ln u - 2 = \ln \dfrac{u}{e^2}$ (c) $\ln u + \ln v - \ln w = \ln \dfrac{uv}{w}$

(b) $3 + \ln v = \ln \dfrac{e^3}{v}$ (d) $\ln 3 + \ln 5 = \ln 8$

Ans: (a) and (c) are valid; (b) and (d) are not.

5. Prove that $\ln(u/v) = \ln u - \ln v$.

Ans: By definition, $e^{\ln(u/v)} = \frac{u}{v}$. But we can also write $\frac{u}{v} = \frac{e^{\ln u}}{e^{\ln v}} = e^{(\ln u - \ln v)}$. Equating the two expressions for $\frac{u}{v}$, we obtain $\ln \frac{u}{v} = \ln u - \ln v$.

EXERCISE 10.4

1. The form of the inverse function of $y = Ae^{rt}$ in (10.17) requires r to be nonzero. What is the meaning of this requirement when viewed in reference to the original exponential function $y = Ae^{rt}$?

Ans: If $r = 0$, then $y = Ae^{rt} = Ae^0 = A$, and the function degenerates into a constant function. The nonzero requirement serves to preclude this contingency.

2. (a) Sketch a graph of the exponential function $y = Ae^{rt}$; indicate the value of the vertical intercept.

(b) Then sketch the graph of the log function $t = \dfrac{\ln y - \ln A}{r}$, and indicate the value of the horizontal intercept.

Ans: The graphs are of the same general shape as in Fig. 10.3; the y intercepts will be A (i.e., $y = A$) for both.

3. Find the inverse function of $y = ab^{ct}$.

Ans: Since $y = ab^{ct}$, we have $\log_b y = \log_b a + ct \log_b b = \log_b a + ct$.

Thus, by solving for t, we get

$t = \frac{\log_b y - \log_b a}{c} \quad (c \neq 0)$

This is the desired inverse function because it expresses t in terms of y.

4. Transform the following functions to their natural exponential forms:
   (a) $y = 8^{3t}$ (c) $y = 5(5)^t$
   (b) $y = 2(7)^{2t}$ (d) $y = 2(15)^{4t}$

Ans:
(a) $a = 1$, $b = 8$, and $c = 3$; thus $r = 3\ln 8$, and $y = e^{(3\ln 8)t}$. We can also write this as $y = e^{6.2385t}$.
(b) $a = 2$, $b = 7$, and $c = 2$; thus $r = 2\ln 7$, and $y = 2e^{(2\ln 7)t}$. We can also write this as $y = 2e^{3.8918t}$.
(c) $a = 5$, $b = 5$, and $c = 1$; thus $r = \ln 5$, and $y = 5e^{(\ln 5)t}$. We can also write this as $y = 5e^{1.6095t}$.
(d) $a = 2$, $b = 15$, and $c = 4$; thus $r = 4\ln 15$, and $y = 2e^{(4\ln 15)t}$. We can also write this as $y = 2e^{10.8324t}$.

5. Transform the following functions to their natural logarithmic forms:
   (a) $t = \log_7 y$ (c) $t = 3\log_{15}(9y)$
   (b) $t = \log_8(3y)$ (d) $t = 2\log_{10} y$

Ans:
(a) $a = 1$, $b = 7$, and $c = 1$; thus $t = \frac{1}{\ln 7}\ln y$ $(= \frac{1}{1.9459}\ln y = 0.5139\ln y)$
(b) $a = 1$, $b = 8$, and $c = 3$; thus $t = \frac{1}{\ln 8}\ln 3y$ $(= \frac{1}{2.0795}\ln 3y = 0.4809\ln 3y)$
(c) $a = 3$, $b = 15$, and $c = 9$; thus $t = \frac{3}{\ln 15}\ln 9y$ $(= \frac{3}{2.7081}\ln 9y = 1.1078\ln 9y)$
(d) $a = 2$, $b = 10$, and $c = 1$; thus $t = \frac{2}{\ln 10}\ln y$ $(= \frac{2}{2.3026}\ln y = 0.8686\ln y)$

6. Find the continuous-compounding nominal interest rate per annum (r) that is equivalent to a discrete-compounding interest rate (i) of

(a) 5 percent per annum, compounded annually.
(b) 5 percent per annum, compounded semiannually.
(c) 6 percent per annum, compounded semiannually.
(d) 6 percent per annum, compounded quarterly.

Ans: The conversion involved is $Ae^{rt} = A(1+\frac{i}{c})^{ct}$, where c represents the number of compoundings per year. Similarly to formula (10.18), we can obtain a general conversion formula $r = c\ln(1+\frac{i}{c})$.

(a) $c=1$, and $i=0.05$; thus $r=\ln 1.05$.
(b) $c=2$, and $i=0.05$; thus $r=2\ln 1.025$.
(c) $c=2$, and $i=0.06$; thus $r=2\ln 1.03$.
(d) $c=4$, and $i=0.06$; thus $r=4\ln 1.015$.

7. (a) In describing Fig. 10.3, the text states that, if the two curves are laid over each other, they show a mirror-image relationship. Where is the "mirror" located?
(b) If we plot a function $f(x)$ and its negative, $-f(x)$, in the same diagram, will the two curves display a mirror-image relationship, too? If so, where is the "mirror" located in this case?
(c) If we plot the graphs of Ae^{rt} and Ae^{-rt} in the same diagram, will the two curves be mirror images of each other? If so, where is the "mirror" located?

Ans:
(a) The $45°$ line drawn through the origin serves as a mirror.
(b) Yes. The horizontal axis is a mirror.
(c) Yes. The horizontal axis is a mirror.

EXERCISE 10.5

1. Find the derivatives of:

(a) $y=e^{2t+4}$ (d) $y=5e^{2-t^2}$ (g) $y=x^2e^{2x}$

(b) $y=e^{1-9t}$ (e) $y=e^{ax^2+bx+c}$ (h) $y=axe^{bx+c}$

(c) $y=e^{t^2+1}$ (f) $y=xe^x$

Ans:

(a) $2e^{2t+4}$

(b) $-9e^{1-7t}$

(c) $2te^{t^2+1}$

(d) $-10te^{2-t^2}$

(e) $(2ax+b)e^{ax^2+bx+c}$

(f) $\frac{dy}{dx} = x\frac{d}{dx}e^x + e^x\frac{dx}{dx} = xe^x + e^x = (x+1)e^x$

(g) $\frac{dy}{dx} = x^2(2e^{2x}) + 2xe^{2x} = 2x(x+1)e^{2x}$

(h) $\frac{dy}{dx} = a(xbe^{bx+c} + e^{bx+c}) = a(bx+1)e^{bx+c}$

2. (a) Verify the derivative in Example 3 by utilizing the equation $\ln(at) = \ln a + \ln t$.

   (b) Verify the result in Example 4 by utilizing the equation $\ln t^c = c\ln t$.

Ans:

(a) $\frac{d}{dt}\ln at = \frac{d}{dt}(\ln a + \ln t) = 0 + \frac{d}{dt}\ln t = \frac{1}{t}$

(b) $\frac{d}{dt}\ln t^c = \frac{d}{dt}c\ln t = c\frac{d}{dt}\ln t = \frac{c}{t}$

3. Find the derivatives of:

   (a) $y = \ln(7t^5)$
   (b) $y = \ln(at^c)$
   (c) $y = \ln(t+19)$
   (d) $y = 5\ln(t+1)^2$
   (e) $y = \ln x - \ln(1+x)$
   (f) $y = \ln[x(1-x)^8]$
   (g) $y = \ln\left(\frac{2x}{1+x}\right)$
   (h) $y = 5x^4\ln x^2$

Ans:

(a) $\frac{dy}{dt} = \frac{35t^4}{7t^5} = \frac{5}{t}$

(b) $\frac{dy}{dt} = \frac{act^{c-1}}{at^c} = \frac{c}{t}$

(c) $\frac{dy}{dt} = \frac{1}{t+99}$

(d) $\frac{dy}{dt} = 5\frac{2(t+1)}{(t+1)^2} = \frac{10}{t+1}$

(e) $\frac{dy}{dx} = \frac{1}{x} - \frac{1}{1+x} = \frac{1}{x(1+x)}$

(f) $\frac{dy}{dx} = \frac{d}{dx}[\ln x + 8\ln(1-x)] = \frac{1}{x} + \frac{-8}{1-x} = \frac{1-9x}{x(1-x)}$

(g) $\frac{dy}{dx} = \frac{d}{dx}[\ln 2x - \ln(1+x)] = \frac{2}{2x} - \frac{1}{1+x} = \frac{1}{x(1+x)}$

(h) $\frac{dy}{dx} = 5x^4 \frac{2x}{x^2} + 20x^3 \ln x^2 = 10x^3(1+2\ln x^2) = 10x^3(1+4\ln x)$

4. Find the derivatives of:

(a) $y = 5^t$ (c) $y = 13^{2t+3}$ (e) $y = \log_2(8x^2+3)$

(b) $y = \log_2(t+1)$ (d) $y = \log_7 7x^2$ (f) $y = x^2 \log_3 x$

Ans:

(a) $\frac{dy}{dt} = 5^t \ln 5$

(b) $\frac{dy}{dt} = \frac{1}{(t+1)\ln 2}$

(c) $\frac{dy}{dt} = 2(13)^{2t+3} \ln 13$

(d) $\frac{dy}{dx} = \frac{14x}{7x^2}\frac{1}{\ln 7} = \frac{2}{x\ln 7}$

(e) $\frac{dy}{dx} = \frac{16x}{(8x^2+3)\ln 2}$

(f) $\frac{dy}{dx} = x^2 \frac{d}{dx}\log_3 x + \log_3 x \frac{d}{dx} x^2 = x^2 \frac{1}{x\ln 3} + (\log_3 x)(2x) = \frac{x}{\ln 3} + 2x\log_3 x$

5. Prove the two formulas in (10.21’).

Ans:

(a) Let $u = f(t)$, so that $du/dt = f'(t)$. Then

$$\frac{d}{dt} b^{f(t)} = \frac{db^u}{dt} = \frac{db^u}{du}\frac{du}{dt} = (b^u \ln b)f'(t) = f'(t)b^{f(t)} \ln b$$

(b) Let $u = f(t)$. Then

$$\frac{d}{dt}\log_b f(t) = \frac{d}{dt}\log_b u = \frac{d}{du}\log_b u \frac{du}{dt} = \frac{1}{u\ln b} f'(t) = \frac{f'(t)}{f(t)}\frac{1}{\ln b}$$

6. Show that the function $V = Ae^{rt}$ (with A, r>0) and the function $A = Ve^{-rt}$ (with V, r>0) are both strictly monotonic, but in opposite directions, and that they are both strictly convex in shape (cf. Exercise 10.2-5).

Ans: For $V = Ae^{rt}$, the first two derivatives are $V' = rAe^{rt} > 0$ and $V'' = r^2Ae^{rt} > 0$. Thus V is strictly increasing at an increasing rate, yielding a strictly convex curve. For $A = Ve^{-rt}$, the first two derivatives are $A' = -rVe^{-rt} < 0$ and $A'' = r^2Ve^{-rt} > 0$. Thus A is strictly decreasing at an increasing rate (with the negative slope taking smaller numerical value as t increases), also yielding a strictly convex curve.

7. Find the derivatives of the following by first taking the natural log of both sides:

(a) $y = \dfrac{3x}{(x+2)(x+4)}$ (b) $y = (x^2+3)e^{x^2+1}$

Ans:

(a) Since $\ln y = \ln 3x - \ln(x+2) - \ln(x+4)$, we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} - \frac{1}{x+2} - \frac{1}{x+4} = \frac{8-x^2}{x(x+2)(x+4)} \text{ and } \frac{dy}{dx} = \frac{8-x^2}{x(x+2)(x+4)}\frac{3x}{(x+2)(x+4)} = \frac{3(8-x^2)}{(x+2)^2(x+4)^2}$$

(b) Since $\ln y = \ln(x^2+3) + x^2 + 1$, we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2+3} + 2x = \frac{2x(x^2+4)}{x^2+3} \text{ and } \frac{dy}{dx} = \frac{2x(x^2+4)}{x^2+3}(x^2+3)e^{x^2+1} = 2x(x^2+4)e^{x^2+1}$$

EXERCISE 10.6

1. If the value of wine grows according to the function $V = Ke^{2\sqrt{t}}$, instead of as in (10.22), how long should the dealer store the wine?

Ans: Since $A = Ke^{2\sqrt{t}-rt}$, we have $\ln A = \ln K + 2\sqrt{t} - rt$. Differentiation with respect to t yields $\frac{1}{A}\frac{dA}{dt} = t^{-1/2} - r$ or $\frac{dA}{dt} = A(t^{-1/2} - r)$

Setting $\frac{dA}{dt} = 0$, we then find: $t^* = 1/r^2$

In the second derivative, $\frac{d^2A}{dt^2} = A\frac{d}{dt}(t^{-1/2} - r) + (t^{-1/2} - r)\frac{dA}{dt}$

The second term vanishes when $\frac{dA}{dt} = 0$. Thus $\frac{d^2A}{dt^2} = -A/2\sqrt{t^3} < 0$, which satisfies the second-order condition for a maximum.

2. Check the second-order condition for the timber-cutting problem.

Ans: $\frac{d^2A}{dt^2} = A\frac{d}{dt}(\frac{\ln 2}{2\sqrt{t}} - r) + (\frac{\ln 2}{2\sqrt{t}} - r)\frac{dA}{dt} = A\frac{d}{dt}(\frac{\ln 2}{2}t^{-1/2} - r) + 0 = \frac{-A\ln 2}{4\sqrt{t^3}} < 0$ [Since $A < 0$ and $\ln 2 > 0$]

Thus the second-order condition is satisfied.

3. As a generalization of the optimization problem illustrated in the present section, show that:

   (a) With any value function $V = f(t)$ and given continuous rate of discount r, the first-order condition for the present value A of V to reach a maximum is that the rate of growth of V be equal to r.

   (b) The second-order sufficient condition for a maximum really amounts to the stipulation that the rate of growth of V be strictly decreasing with time.

Ans:

(a) Since $A = Ve^{-rt} = f(t)e^{-rt}$, we have $\ln A = \ln f(t) - rt$, and

$$\frac{1}{A}\frac{dA}{dt} = \frac{f'(t)}{f(t)} - r = r_v - r \quad \text{or} \quad \frac{dA}{dt} = A(r_v - r)$$

Inasmuch as A is nonzero, $dA/dt = 0$ if and only if $r_v = r$.

(b) The second derivative is $\frac{d^2A}{dt^2} = A\frac{d}{dt}\frac{f'(t)}{f(t)} = A\frac{d}{dt}r_v < 0$ iff $\frac{d}{dt}r_v < 0$

4. Analyze the comparative statics of the wine-storage problem.

Ans: The value of t^* depends only on the parameter r. Since $\frac{dt^*}{dr} = \frac{d}{dr}\frac{1}{4}r^{-2} = -\frac{1}{2r^3} < 0$ a higher interest rate means a smaller t^* (an earlier optimal time of sale).

EXERCISE 10.7

1. Find the instantaneous rate of growth:

   (a) $y = 5t^2$ (c) $y = ab^t$ (e) $y = t/3^t$

   (b) $y = at^c$ (d) $y = 2^t(t^2)$

Ans:

(a) $\ln y = \ln 5 + 2\ln t$; thus $r_y = \frac{d}{dt}\ln y = \frac{2}{t}$.

(b) $\ln y = \ln a + c\ln t$; thus $r_y = c/t$.

(c) $\ln y = \ln a + t\ln b$; thus $r_y = \ln b$.

(d) Let $u = 2^t$ and $v = t^2$. Then $r_u = \ln 2$, and $r_v = 2/t$. Thus $r_y = r_u + r_v = \ln 2 + 2/t$. Alternatively, we can write $\ln y = t\ln 2 + 2\ln t$; thus $r_y = \frac{d}{dt}\ln y = \ln 2 = \frac{2}{t}$.

(e) Let $u = t$ and $v = 3^t$. Then $r_u = \frac{d(\ln u)}{dt} = \frac{d(\ln t)}{dt} = \frac{1}{t}$, and $r_v = \frac{d(\ln v)}{dt} = \frac{d(\ln 3^t)}{dt} = \frac{d(t\ln c)}{dt} = \ln 3$. Consequently, $r_y = r_u + r_v = \frac{1}{t} - \ln 3$. Alternatively, we can write $\ln y = \ln t - t\ln 3$; thus $r_y = \frac{d}{dt}\ln y = \frac{1}{t} - \ln 3$

2. If population grows according to the function $H = H_0(2)^{bt}$ and consumption by the function $C = C_0 e^{at}$, find the rates of growth of population, of consumption, and of per capita consumption by using the natural log.

Ans: $\ln H = \ln H_0 + bt\ln 2$; thus $r_H = b\ln 2$. Similarly, $\ln C = \ln C_0 + at\ln e$; thus $r_C = a\ln e = a$. It follows that $r_{(C/H)} = r_C - r_H = a - b\ln 2$.

3. If y is related to x by $y = x^k$, how will the rates of growth r_y and r_x, be related?

Ans: Taking log, we get $\ln y = k\ln x$. Differentiating with respect to t, we then obtain $r_y = kr_x$.

4. Prove that if $y = u/v$, where $u = f(t)$ and $v = g(t)$, then the rate of growth of y will be $r_y = r_u - r_v$, as shown in (10.25).

Ans: $y = \frac{u}{v}$ implies $\ln y = \ln u - \ln v$; it follows that

$r_y = \frac{d}{dt}\ln y = \frac{d}{dt}\ln u - \frac{d}{dt}\ln v = r_u - r_v$.

5. The real income y is defined as the nominal income Y deflated by the price level P. How is r_y (for real income) related to r_Y (for nominal income)?

Ans: By definition, $y = Y/P$. Taking the natural log, we have $\ln y = \ln Y - \ln P$. Differentiation of $\ln y$ with respect to time t yields $\frac{d}{dt}\ln y = \frac{d}{dt}\ln Y - \frac{d}{dt}\ln P$. Which means $r_y = r_Y - r_P$ were r_P is the rate of inflation.

6. Prove the rate-of-growth rule (10.27).

Ans: $z = u - v$ implies $\ln z = \ln(u - v)$; thus

$r_z = \frac{d}{dt}\ln z = \frac{d}{dt}\ln(u - v) = \frac{1}{u-v}\frac{d}{dt}\ln(u - v) = \frac{1}{u-v}\frac{d}{dt}\ln[f'(t) - g'(t)] = \frac{1}{u-v}(ur_u - vr_v)$

7. Given the demand function $Q_d = k/P^n$, where k and n are positive constants, find the point elasticity of demand ε_d by using (10.28) (cf. Exercise 8.1-4).

Ans: $\ln Q_d = \ln k - n\ln P$. Thus, by (10.28), $\varepsilon_d = -n$, and $|\varepsilon_d| = n$.

8. (a) Given $y = wz$, where $w = g(x)$ and $z = h(x)$, establish that $\varepsilon_{yx} = \varepsilon_{wx} + \varepsilon_{zx}$.

(b) Given $y = u/v$, where $u = G(x)$ and $v = H(x)$, establish that $\varepsilon_{yx} = \varepsilon_{ux} - \varepsilon_{vx}$.

Ans:

(a) Since $\ln y = \ln w + \ln z$, we have $\varepsilon_{yx} = \frac{d(\ln y)}{d(\ln x)} = \frac{d(\ln w)}{d(\ln x)} + \frac{d(\ln z)}{d(\ln x)} = \varepsilon_{wx} + \varepsilon_{zx}$

(b) Since $\ln y = \ln u - \ln v$, we have $\varepsilon_{yx} = \frac{d(\ln y)}{d(\ln x)} = \frac{d(\ln u)}{d(\ln x)} - \frac{d(\ln v)}{d(\ln x)} = \varepsilon_{ux} + \varepsilon_{vx}$

9. Given $y = f(x)$, show that the derivative $d(\log_b y)/d(\log_b x)$—log to base b rather than e—also measures the point elasticity ε_{yx}.

Ans: Let $u = \log_b y$, and $v = \log_b x$ (implying that $x = b^v$). Then $\frac{du}{dv} = \frac{du}{dy}\frac{dy}{dx}\frac{dx}{dv} = \frac{1}{y}(\log_b e)\frac{dy}{dx} b^v \ln b$. Since $\log_b e = \frac{1}{\ln b}$, and since $b^v = x$, we have $\frac{du}{dv} = \frac{x}{y}\frac{dy}{dx} = \varepsilon_{yx}$.

10. Show that, if the demand for money M_d is a function of the national income $Y = Y(t)$ and the interest rate $i = i(t)$, the rate of growth of M_d can be expressed as a weighted sum of r_Y and r_i,

$$r_{M_d} = \varepsilon_{M_d Y} r_Y + \varepsilon_{M_d i} r_i$$

where the weights are the elasticities of M_d with respect to Y and i, respectively.

Ans: Since $M_d = f[Y(t), i(t)]$, we can write the total derivative $\frac{dM_d}{dt} = f_y \frac{dy}{dt} + f_i \frac{di}{dt}$. Thus the rate of growth of M_d is

$$r_{M_d} = \frac{dM_d/dt}{M_d} = \frac{f_Y}{f}\frac{dY}{dt} + \frac{f_i}{f}\frac{di}{dt} = \frac{f_Y}{f}\frac{Y}{Y}\frac{dY}{dt} + \frac{f_i}{f}\frac{i}{i}\frac{di}{dt} = \frac{f_Y Y}{f}\left(\frac{1}{Y}\frac{dY}{dt}\right) + \frac{f_i i}{f}\left(\frac{1}{i}\frac{di}{dt}\right) = \varepsilon_{M_d Y} r_Y + \varepsilon_{M_d i} r_i \quad .$$

Alternatively, using logarithms, we may write $r_{M_d} = \frac{d}{dt}\ln M_d = \frac{1}{M_d}\frac{d}{dt}M_d$, but this then leads us back to the preceding process.

11. Given the production function $Q = F(K, L)$, find a general expression for the rate of growth of Q in terms of the rates of growth of K and L.

Ans: By the same procedure used in 9 above, we can find that $r_Q = \varepsilon_{QK} r_K + \varepsilon_{QL} r_L$.