Chiang/Wainwright: Fundamental Methods of Mathematical Economics

CHAPTER 9

EXERCISE 9.2

1. Find the stationary values of the following (check whether they are relative maxima or minima or inflection points), assuming the domain to be the set of all real numbers:
   (a) $y = -2x^2 + 8x + 7$ (b) $y = 5x^2 + x$ (c) $y = 3x^2 + 3$
   (d) $y = 3x^2 - 6x + 2$

Ans:
(a) $f'(x) = -4x + 8 = 0$ iff $x = 2$; the stationary value $f(2) = 15$ is a relative maximum.
(b) $f'(x) = 10x + 1 = 0$ iff $x = -1/10$; the stationary value $f(-1/10) = -1/20$ is a relative minimum.
(c) $f'(x) = 6x = 0$ iff $x = 0$; the stationary value $f(0) = 3$ is a relative minimum.

2. Find the stationary values of the following (check whether they are relative maxima or minima or inflection points), assuming the domain the be the interval $[0, \infty)$:
   (a) $y = x^3 - 3x + 5$
   (b) $y = \frac{1}{3}x^3 - x^2 + x + 10$
   (c) $y = -x^3 + 4.5x^2 - 6x + 6$

Ans:
(a) Setting $f'(x) = 3x^2 - 3 = 0$ yields two critical values, 1 and –1. The latter is outside the domain; the former leads to $f(1) = 3$, a relative minimum.
(b) The only critical value is $x^* = 1$; $f(1) = 10\frac{1}{3}$ is a point of inflection.
(c) Setting $f'(x) = -3x^2 + 9x - 6 = 0$ yields two critical values, 1 and 2; $f(1) = 3.5$ is a relative minimum but $f(2) = 4$ is a relative maximum.

3. Show that the function $y = x + \frac{1}{x}$ (with $x \neq 0$) has two relative extrema, one a maximum and the other a minimum. Is the “minimum” larger or smaller than the “maximum”? How is this paradoxical result possible?

Ans: When $x = 1$, we have $y = 2$ (a minimum); when $x = -1$, we have

$y = -2$ (a maximum). These are in the nature of relative extrema, thus a minimum can exceed a maximum.

4. Let $T = \phi(x)$ be a total function (e.g., total product or total cost):
   (a) Write out the expressions for the marginal function M and the average function A.
   (b) Show that, when A reaches a relative extremum, M and A must have the same value.
   (c) What general principle does this suggest for the drawing of a marginal curve and an average curve in the same diagram?
   (d) What can you conclude about the elasticity of the total function T at the point where A reaches an extreme value?

Ans:

(a) $M = \phi'(x)$, $A = \phi(x)/x$

(b) When A reaches a relative extremum, we must have

$$\frac{dA}{dx} = \frac{1}{x^2}[x\phi'(x) - \phi(x)] = 0$$

This occurs only when $x\phi'(x) = \phi(x)$, that is, only when $\phi'(x) = \phi(x)/x$, or only when $M = A$.

(c) The marginal and average curves must intersect when the latter reaches a peak or a trough.

(d) $\varepsilon = \frac{M}{A} = 1$ when $M = A$.

EXERCISE 9.3

1. Find the second and third derivatives of the following functions:

   (a) $ax^2 + bx + c$ (c) $\frac{3x}{1-x}$ $(x \neq 1)$

   (b) $7x^4 - 3x - 4$ (d) $\frac{1+x}{1-x}$ $(x \neq 1)$

Ans:

(a) $f'(x) = 2ax + b$; $f''(x) = 2a$; $f'''(x) = 0$

(b) $f'(x) = 28x^3 - 3$; $f''(x) = 84x^2$; $f'''(x) = 168x$

(c) $f'(x) = 3(1-x)^{-2}$; $f''(x) = 6(1-x)^{-3}$; $f'''(x) = 18(1-x)^{-4}$

(d) $f'(x) = 2(1-x)^{-2}$; $f''(x) = 4(1-x)^{-3}$; $f'''(x) = 12(1-x)^{-4}$

2. Which of the following quadratic functions are strictly convex?

(a) $y = 9x^2 - 4x + 8$ (c) $u = 9 - 2x^2$

(b) $w = -3x^2 + 39$ (d) $v = 8 - 5x + x^2$

Ans: (a) and (d)

3. Draw (a) a concave curve which is not strictly concave, and (b) a curve which qualifies simultaneously as a concave curve and a convex curve.

Ans:

(a) An example is a modified version of the curve in Fig. 9.5a, with the arc AB replaced by a line segment AB.

(b) A straight line.

4. Given the function $y = a - \frac{b}{c+x}$ $(a, b, c > 0; x \geq 0)$, determine the general shape of its graph by examining (a) its first and second derivatives, (b) its vertical intercept, and (c) the limit of y as x tends to infinity. If this function is to be used as a consumption function, how should the parameters be restricted in order to make it economically sensible?

Ans: Since $dy/dx = b/(c+x)^2 > 0$, and $d^2y/dx^2 = -2b/(c+x)^3 < 0$, the curve must show y increasing at a decreasing rate. The vertical intercept (where $x = 0$) is $a - \frac{b}{c}$. When x approaches infinity, y tends to the value a, which gives a horizontal asymptote. Thus the range of the function is the interval $[a - \frac{b}{c}, a)$.

To use it as a consumption function, we should stipulate that:

$a > \frac{b}{c}$ [so that consumption is positive at zero income]

$b > c^2$ [so that $MPC = dy/dx$ is a positive fraction throughout]

5. Draw the graph of a function $f(x)$ such that $f'(x) \equiv 0$, and the graph of a function $g(x)$ such that $g'(3) = 0$. Summarize in one sentence the essential difference between $f(x)$ and $g(x)$ in terms of the concept of stationary point.

Ans: The function $f(x)$ plots as a straight line, and $g(x)$ plots as a curve with either a peak or a bottom or an inflection point at $x = 3$. In terms of stationary points, every point on $f(x)$ is a stationary point, but the only stationary point on $g(x)$ we know of is at $x = 3$.

6. A preson who is neither risk-averse nor risk-loving (indifferent toward a fair game) is said to be “risk-neutral.”
   (a) What kind of utility function would you use to characterize such a person?
   (b) Using the die-throwing game detailed in the text, describe the relationship between U($15) and EU for the risk-neutral person.

Ans:

(a) The utility function should have $f(0) - 0, f'(x) > 0$, and $f''(x) = 0$ for all x. It plots as an upward-sloping straight line emanating from the point of origin.

(b) In the present case, the MN line segment would coincide with the utility curve. Thus points A and B lie on top of each other, and $U(15) = EU$.

EXERCISE 9.4

1. Find the relative maxima and minima of y by the second-derivative test:

   (a) $y = -2x^2 + 8x + 25$ (c) $y = \frac{1}{3}x^3 - 3x^2 + 5x + 3$

   (b) $y = x^3 + 6x^2 + 9$ (d) $y = \dfrac{2x}{1-2x} \quad (x \neq \dfrac{1}{2})$

Ans:

(a) $f'(x) = -4x + 8$; $f''(x) = -4$. The critical value is $x^* = 2$; the stationary value $f(2) = 33$ is a maximum.

(b) $f'(x) = 3x^2 + 12x$; $f''(x) = 6x + 12$. The critical value is 0 and –4. $f(0) = 9$ is a minimum, because $f''(0) = 12 > 0$, but $f(-4) = 41$ is a maximum, because $f''(-4) = -12 < 0$.

(c) $f'(x) = x^2 - 6x + 5$; $f''(x) = 2x - 6$. The critical value is 1 and 5. $f(1) = 5\dfrac{1}{3}$ is a maximum, because $f''(1) = -4$, but $f(5) = -5\dfrac{1}{3}$ is a minimum, because $f''(5) = 4$.

(d) $f'(x) = 2/(1-2x)^2 \neq 0$ for any value of x; there exists no relative extremum.

2. Mr. Greenthumb wishes to mark out a rectangular flower bed, using a wall of his house as one side of the rectangle. The other three sides are to be marked by wire netting, of which he has only 64 ft available. What are the length L and width W of the rectangle that would give him the largest possible planting area? How do you make sure that your answer gives the largest, not the smallest area?

Ans: Excluding the wall side, the other three sides must satisfy $L + 2W = 64\text{ft}$, or $L = 64 - 2W$. The area is therefore

$$A = WL = W(64 - 2W) = 64W - 2W^2$$

To maximize A, it is necessary that $dA/dW = 64 - 4W = 0$, which can occur only when $W = 16$. Thus

$$W^* = 16\text{ft} \qquad L^* = 64 - 2W^* = 32\text{ft} \qquad A^* = WL = 512\text{ft}^2$$

Inasmuch as $d^2A/dW^2 = -4$ is negative, A^* is a maximum.

3. A firm has the following total-cost and demand functions:

$$C = \tfrac{1}{3}Q^3 - 7Q^2 + 111Q + 50$$

$$Q = 100 - P$$

(a) Does the total-cost function satisfy the coefficient restrictions of (9.5)?
(b) Write out the total-revenue function R in terms of Q.
(c) Formulate the total-profit function π in terms of Q.
(d) Find the profit-maximizing level of output Q^*.
(e) What is the maximum profit?

Ans:

(a) Yes.

(b) From the demand function, we first get the AR function $P = 100 - Q$. Then we have $R = PQ = (100 - Q)Q = 100Q - Q^2$.

(c) $\pi = R - C = -\frac{1}{3}Q^3 + 6Q^2 - 11Q - 50$

(d) Setting $d\pi/dQ = -Q^2 + 12Q - 11 = 0$ yields two critical values 1 and 11. Only $Q^* = 11$ gives a maximum profit.

(e) Maximum profit $= 111\frac{1}{3}$.

4. If coefficient b in (9.3) were to take a zero value, what would happen to the marginal-cost and total-cost curves?

Ans: If $b = 0$, then the MC-minimizing output level becomes $Q^* = -\frac{b}{3a} = 0$. With its minimum at zero output. The MC curve must be upward-sloping throughout. Since the increasing segment of MC is associated with the convex segment of the C curve, $b = 0$ implies that the C curve will be convex throughout.

5. A quadratic profit function $\pi(Q) = hQ^2 + jQ + k$ is to be used to reflect the following assumptions:
   (a) If nothing is produced, the profit will be negative (because of fixed costs).
   (b) The profit function is strictly concave.
   (c) The maximum profit occurs at a positive output level Q^*.

   What parameter restrictions are called for?

Ans:

(a) The first assumption means $\pi(0) < 0$. Since $\pi(0) = k$, we need the restriction $k < 0$.

(b) Strict concavity means $\pi''(Q) < 0$. Since $\pi''(Q) = 2h$, we should have $h < 0$.

(c) The third assumption means $\pi'(Q^*) = 0$, or $2hQ^* + j = 0$. Since $Q^* = -j/2h$, and since $h < 0$, the positivity of Q^* requires that $j > 0$.

6. A purely competitive firm has a single variable input L (labor), with the wage rate W_0 per period. Its fixed inputs cost the firm a total of F dollars per period. The price of the product is P_0.
   (a) Write the production function, revenue function, cost function, and profit function of the firm.
   (b) What is the first-order condition for profit maximization? Give this condition an economic interpretation.
   (c) What economic circumstances would ensure that profit is maximized rather than minimized?

Ans:

(a) $Q = f(L)$; $R = P_0 Q = P_0 f(L)$; $C = W_0 L + F$; $\pi = R - C = P_0 f(L) - W_0 L - F$

(b) $d\pi/dL = P_0 f'(L) - W_0 = 0$, or $P_0 f'(L) = W_0$. The value of marginal product must be equated to the wage rate.

(c) $d^2\pi/dL^2 = P_0 f''(L)$. If $f''(L) < 0$ (diminishing MPP_L), then we can be sure that profit is maximized by L^*.

7. Use the following procedure to verify that the AR curve in Example 4 is negatively sloped:
   (a) Denote the slope of AR by S. Write an expression for S.
   (b) Find the maximum value of S, S_{max}, by using the second-derivative test.
   (c) Then deduce from the value of S_{max} that the AR curve is negatively sloped throughout.

Ans:

(a) $S = \frac{d}{dQ}AR = -23 + 2.2Q - 0.054Q^2$

(b) $\frac{dS}{dQ} = 2.2 - 0.108Q = 0$ at $Q^* = 20.37$ (approximately); since $\frac{d^2S}{dQ^2} = -0.108 < 0$, Q^* will maximize S.

$S_{max} = S\big|_{Q=Q^*} = -23 + 2.2(20.37) - 0.054(20.37)^2 = -0.59$ (approximately).

(c) Since S_{max} is negative, all S values must be negative.

EXERCISE 9.5

1. Find the value of the following factorial expressions:

   (a) 5! (c) $\frac{4!}{3!}$ (e) $\frac{(n+2)!}{n!}$

   (b) 8! (d) $\frac{6!}{4!}$

Ans: (a) 120 (b) 40320 (c) $\frac{4(3!)}{3!} = 4$ (d) $\frac{(6)(5)(4!)}{4!} = 6 \cdot 5 = 30$

(e) $\frac{(n+2)(n+1)n!}{n!} = (n+2)(n+1)$

2. Find the first five terms of the Maclaurin series (i.e., choose $n = 4$ and let $x_0 = 0$) for:

   (a) $\phi(x) = \frac{1}{1-x}$ (b) $\phi(x) = \frac{1-x}{1+x}$

Ans:

(a) $\phi(x) = (1-x)^{-1}$ so that $\phi(0) = 1$

$\phi'(x) = (1-x)^{-2}$ $\phi'(0) = 1$

$\phi''(x) = 2(1-x)^{-3}$ $\phi''(0) = 2$

$\phi'''(x) = 6(1-x)^{-4}$ $\phi'''(0) = 6$

$\phi^{(4)}(x) = 24(1-x)^{-5}$ $\phi^{(4)}(0) = 24$

Thus, according to (9.14), the first five terms are $1 + x + x^2 + x^3 + x^4$

(b) $\phi(x) = (1-x)/(1+x)$ so that $\phi(0) = 1$

$\phi'(x) = -2(1-x)^{-2}$ $\phi'(0) = -2$

$\phi''(x) = 4(1+x)^{-3}$ $\phi''(0) = 4$

$\phi'''(x) = -12(1+x)^{-4}$ $\phi'''(0) = -12$

$\phi^{(4)}(x) = 48(1+x)^{-5}$ $\phi^{(4)}(0) = 48$

Thus, by (9.14), the first five terms are $1 - 2x + 2x^2 - 2x^3 + 2x^4$

3. Find the Taylor series with $n = 4$ and $x_0 = -2$, for the two functions in Prob. 2.

Ans:

(a) $\phi(-2) = 1/3$, $\phi'(-2) = 1/9$, $\phi''(-2) = 2/27$, $\phi'''(-2) = 6/81$, and $\phi^{(4)}(-2) = 24/243$. Thus, by (9.14),

$$\phi(x) = \frac{1}{3} + \frac{1}{9}(x+2) + \frac{1}{27}(x+2)^2 + \frac{1}{81}(x+2)^3 + \frac{1}{243}(x+2)^4 + R_4$$

$$= \frac{1}{243}(211 + 131x + 51x^2 + 11x^3 + x^4) + R_4$$

(b) $\phi(-2) = -3, \phi'(-2) = -2, \phi''(-2) = -4, \phi'''(-2) = -12$, and $\phi^{(4)}(-2) = -48$. Thus, by (9.14),

$$\phi(x) = -3 - 2(x+2) - 2(x+2)^2 - 2(x+2)^3 - 2(x+2)^4 + R_4$$

$$= -63 - 98x - 62x^2 - 18x^3 - 2x^4 + R_4$$

4. On the basis of Taylor's formula with the Lagrange form of the remainder [see (9.14) and (9.15)], show that at the point of expansion ($x = x_0$) the Taylor series will always give exactly the value of the function at that point, $\phi(x_0)$, not merely an approximation.

Ans: When $x = x_0$, all the terms on the right of (9.14) except the first one will drop out (including R_n), leaving the result $\phi(x) = \phi(x_0)$.

EXERCISE 9.6

1. Find the stationary values of the following functions:

   (a) $y = x^3$ (b) $y = -x^4$ (c) $y = x^6 + 5$

   Determine by the Nth-derivative test whether they represent relative maxima, relative minima, or inflection points.

Ans:

(a) $f'(x) = 3x^2 = 0$ only when $x = 0$, thus $f(0) = 0$ is t he only stationary value. The first nonzero derivative value is $f'''(0) = 6$; so $f(0)$ is an inflection point.

(b) $f'(x) = -4x^3 = 0$ only when $x = 0$. The stationary value $f(0) = 0$ is a relative maximum because the first nonzero derivative value is $f^{(4)}(0) = -24$.

(c) $f'(x) = 6x^5 = 0$ only when $x = 0$. The stationary value $f(0) = 5$ is a relative minimum since the first nonzero derivative value is $f^{(6)}(0) = 720$.

2. Find the stationary values of the following functions:

   (a) $y = (x-1)^3 + 16$ (c) $y = (3-x)^6 + 7$

   (b) $y = (x-2)^4$ (d) $y = (5-2x)^4 + 8$

   Use the Nth-derivative test to determine the exact nature of these stationary values.

Ans:

(a) $f'(x) = 3(x-1)^2 = 0$ only when $x = 1$. The first nonzero derivative value is $f'''(1) = 6$. Thus the stationary value $f(1) = 16$ is associated with an inflection point.

(b) $f'(x) = 4(x-2)^3 = 0$ only when $x = 2$. Since the first nonzero derivative value is $f^{(4)}(2) = 24$, the stationary value $f(2) = 0$ is a relative minimum.

(c) $f'(x) = -6(3-x)^5 = 0$ only when $x = 3$. Since the first nonzero derivative value is $f^{(6)}(3) = 720$, the stationary value $f(3) = 7$ is a relative minimum.

(d) $f'(x) = -8(5-2x)^3 = 0$ only when $x = 2.5$. Since the first nonzero derivative value is $f^{(4)}(2.5) = 384$, the stationary value $f(2.5) = 8$ is a relative minimum.